BOUND OF MEAN, VARIANCE, CO-VARIANCE AND CORRELATIONS FOR ORDERED STATISTICS

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

Master of Philosophy
in
Statistics

By

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ALIGARH 202002, INDIA

2011

Dedicated

To

My Parents

ACKNOWLEDGEMENT

All praises due to Almighty, the most Beneficent, the most Merciful, who bestowed upon me the courage, patience and strength to embark upon this work and carry it to its completion and admires to his last and great Prophet (PBUH), whose moments of life provided me with the strength to run fast on the path of success.

*It is good fortune and a matter of pride and privilege for me to have the esteemed supervision of **Prof. Abdul Hamid Khan,** Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, who has inculcated in me the interest and inspiration to undertake research in the field of order statistics. It is only his personal influence, expert guidance and boundless support that enabled me to complete the work in the present form. I consider it my pleasant duty to express my deepest gratitude to him for his incessant support and never failing inspiration. I was providential to get a lifetime opportunity to work under his knowledgeable supervision.*

*I place on record my special sense of gratitude to **Prof. I.A. Khan,** Chairman, Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh for imparting me necessary facilities in the department and for extending his kind cooperation in every stage of the pursuit of this dissertation.*

*I would like to thank **Prof. M.J. Ahsan , Prof. M. Yaqub, Prof. M.M. Khalid, Prof. Abdul Bari, Prof. A.A. Khan, Prof. Q.M. Ali, Dr. M. Arshad, Dr. R.U. Khan, Dr. Haseeb Athar, Dr. M. Faizan, Mr. S.S. Hasan, Dr. Kamalullah, Dr. S. Javaid, and Dr. Z. Anwar** for their moral support and wholehearted cooperation. They have been a source of inspiration during my academic endeavor.*

*Words cannot express my deepest thanks to my friend **Saima Iqrar** who stood by me in all the sweet and sour moments and was always there for me. Sincere thanks to my friends **Ummatul Fatima** and **Farha Saleem** for their wholehearted support and encouragement.*

I would like to acknowledge my overwhelming gratitude to my research colleagues ***Ms. Benazir Zia, Ms Zoha Qayyum, Ms. Shazia Zarrin, Mrs. Zehra Sherwani, Ms. Farha Naaz, Mrs. Sanghmitra Sharma, Ms. Sana Iftekhar, Ms. Anamika Kulshrestha, Ms Khushboo Goyal, Mr. Devendra Kumar, Mr. M. Izhar Khan, Mr. Imtiyaz Ahmad Shah, Mr Shamsher Khan and Mr. Zuber Akhter*** *who have always been a source of encouragement for me. They helped me a lot during my work.*

Thanks are also due to all non-teaching staff members of the department for their help and cooperation.

I would like to express my deep sense of indebtedness to my parents, who have been dreaming for quite a long time for my success. They have done everything possible to see me at this place. They always helped me in good and bad times alike to keep me focused towards my goal. The completion of this dissertation would have been rather impossible without their wholehearted support, encouragement and sacrifices during the tough and easier phases of this project. I feel shortage of words to pay thanks to them. I would also like to express my special thanks to my sister ***Filzah-e-noor*** *and brothers* ***Mazhar Noor, Athar Noor*** *and* ***Mohammad Danish Noor*** *for their best wishes.*

I would like to take this opportunity to express my gracious concern towards all those people who have helped in various ways throughout the development stages of this dissertation.

Finally, I would like to thank ***The University Grant Commission*** *for providing me with the fellowship.*

Date: 10.5.11

(Zubdah-e-noor)

CONTENTS

PREFACE

The present dissertation entitled **"Bound of Mean, Variance, Co-variance and Correlations for Ordered Statistics"** is a brief collection of the work done so far on the subject. I have tried my best to include sufficient and relevant materials in the systematic way, which are contained in five chapters.

Chapters I is introductory in nature in which some concepts which may be helpful to grasp the ideas contained in the remaining chapters are discussed.

Chapter II consists of some recurrence relations for single moments for some specific continuous truncated distributions. *viz.* Weibull, exponential, Pareto, power function, Cauchy, Burr, log-logistic, gamma and beta distribution.

In chapter III recurrence relations for product moments for the specific, continuous truncated distributions, namely, Weibull, exponential, Pareto, power function, Cauchy, Burr and log-logistic.

Chapter IV embodies a number of general approaches giving bounds and approximations to the expectation of order statistics.

Chapter V presents some developments in the field of bounds of variance, covariance and correlation of order statistics.

In the end, a comprehensive list of references referred into this dissertation is given.

Chapter I

BASIC CONCEPTS AND PRELIMINARIES

1. Order statistics

Suppose that $X_1, X_2, \cdots, X_n$ is a random sample of size n from a continuous distribution. Let they be arranged in ascending order of their magnitude, then the smallest of $X_i's$ is denoted by $X_{1:n}$, the second smallest is denoted by $X_{2:n}$ and finally, the largest is denoted by $X_{n:n}$. Thus it becomes $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$, where $X_{r:n}$ is the r^{th} order statistic from a sample of size n.

A remarkably large body of literature has been devoted to the study of order statistics. Developments through the early 1960's were synthesized in the volume edited by Sarhan and Greenberg (1962), which, because of it's numerous tables, retains its usefulness even today. Harter (1978-1992) has prepared an eight volume annotated bibliography of order statistics with an excess of 4700 entries. The book by David and Nagaraja (2003) is an excellent reference book. Fascinating related odysseys are described in Galambos (1978), Harter (1970), Khan *et al.* (1983 a, b), Barnett and Lewis (1984), Castillo (1988), Balakrishnan and Cohen (1991) and Arnold *et al.* (1992).

In nonparametric problems, it is noted that order statistics are playing a vital role. For this reason, in the beginning [e.g. Scheffe (1943)], order statistics were also considered as a part of nonparametric statistics and one was simply concerned with the properties of a random sample whose observations were arranged in ascending or descending order of magnitude.

Order statistics are particularly useful in nonparametric statistics, because of the properties of an important device called the "probability integral transformation" according to which the distribution of a distribution function is distribution free. Order statistics and functions of order statistics play an important role in statistical theory and methodology. Floods, droughts, longevity, breaking strength, aeronautics, oceanography, life span of humans and organism, components and devices of various kinds can all be studied by the theory of extreme values. The range is widely used, particularly in statistical quality control, as an estimate of σ. Many short-cut tests have been based on the range and other order statistics. In dealing with small samples the studentized ranges are useful in a variety of ways. Apart from supplying the basis of many of quick tests, it plays a key role in procedures for ranking "treatment" means in the analysis of variance situation. By applying the Gauss-Markov theorem of least squares, it is possible to use linear functions of order statistics for estimating the parameters of distribution functions. This application is very useful particularly when some of the observations in the sample have been "censored" since in that case standard methods of estimation tend to become laborious or otherwise unsatisfactory (Lloyd, 1952). Life tests provide an ideal illustration of the advantage of order statistics in censored data. Since such an experiment may take a long time to complete, it is often advantageous to stop after failure of the first r out of n similar items under test.

2. Recurrence Relations and Identities of Order Statistics

Order statistics and their moments have been of great interest from the turn of this century since Galton (1902) and Pearson (1902) studied the distribution of the difference of two successive order statistics. The moments of order statistics assume considerable and fetching importance in the statistical literature and have been numerically tabulated extensively for several distributions. For example, one can refer to Arnold and Balakrishnan (1989), Arnold *et al.* (1992) and David and Nagaraja (2003) for a detailed list of these tables. Many authors have investigated and derived several recurrence relations and identities satisfied by single as well as product moments of order statistics primarily to reduce the amount of direct computations. However, one could list the following three main reasons why these recurrence relations and identities for the moments of order statistics are important:

i) They reduce the amount of direct computations considerably,

ii) They usefully express the higher order moments of order statistics in terms of the lower order moments and hence make the evaluation of higher order moments easy,

iii) They are very useful in checking the computation of the moments of order statistics.

Shah (1966) obtained moments of order statistics from logistic distribution.

Krishnaiah and Rizvi (1967) extended the results of Gupta (1960) for gamma distribution with any positive shape parameter. Young (1971)

established a very simple relation between moments of order statistics of the symmetrical inverse multinomial distribution and the moments of order statistics of independent standardized gamma variables with integer parameter λ.

Joshi (1978) obtained some recurrence relations between the moments of order statistics from exponential and right truncated exponential distributions and later Joshi (1979 a, b) obtained similar recurrence relations for the moments of order statistics from doubly truncated exponential distribution. Joshi (1982) also obtained some recurrence relations for mixed moments of order statistics from exponential and truncated exponential distributions.

Balakrishnan and Joshi (1983 a, b, 1984) obtained recurrence relations for single and product moments of order statistics from symmetrically truncated logistic distribution and doubly truncated exponential distributions.

Joshi and Balakrishnan (1981) obtained an identity for the moments of normal order statistics and showed their applications.

Balakrishnan and Joshi (1982) obtained the recurrence relations for doubly truncated Pareto distribution.

Khan *et al.* (1983 a) have obtained general results for finding the k^{th} moments of order statistics without considering any particular distribution. These results were utilized to obtain recurrence relations for doubly truncated and non-truncated distributions. The examples considered were Weibull, exponential, Pareto, power function, Cauchy, logistic, gamma and beta distribution. Also Khan *et al.* (1983 b) established general results for obtaining the product moments of the j^{th} power of the r^{th} order statistics and the k^{th} power of the s^{th} order statistics from an arbitrary continuous distribution. Then they utilized these results to determine the recurrence relations between product moments of some doubly truncated and non-truncated distributions, viz. Weibull, exponential, Pareto, power function and Cauchy.

Balakrishnan (1985) established some recurrence relations for the single and product moments from half logistic distribution. Balakrishnan and Aggarwala (1996) obtained the relationship for moments of order statistics from the right truncated generalized half logistic distribution.

Balakrishnan *et al.* (1987) derived recurrence relations for product moments of order statistics from log-logistic distribution. Balakrishnan and Malik (1987) obtained relations for moments of order statistics from truncated log-*logistic distribution.* Ali *and* Khan *(1987) obtained the recurrence relations* between moments of order statistics for log- logistic distribution.

Balakrishnan and Kocherlakota (1986) and Al-Shboul and Khan (1989) obtained moments of order statistics for doubly truncated log-logistic distribution.

Khan *et al.* (1984) obtained the inverse moments of order statistics from Weibull distribution whereas Ali and Khan (1996) obtained the ratio and product moments of order statistics from Weibull and exponential distribution. Unifying earlier results Khan and Athar (2000) established relation for ratio and product moments of order statistics from doubly truncated Weibull distribution.

Ali and Khan (1995) have obtained ratio and product moment of two order statistics of different order from Burr distribution. Further they have deduced the moments and inverse moments for single order statistics from the product moments.

Ali and Khan (1997) established the recurrence relations for the expectation of a function of single order statistics from a general class of distribution. Further Ali and Khan (1998) also established the recurrence relations for the expected values of certain functions of two order statistics. Saran and Pushkarana (1999) established the recurrence relation for single and product moments of order statistics from doubly truncated generalized exponential distribution. Related results may also be found in Khan *et al.* (1987) and Khan and Abu-Salih (1988).

It may be noted here that many of the above mentioned results for the moments of order statistics from an arbitrary continuous distributions have been listed in a survey article by Malik *et al.* (1988) whereas Balakrishnan *et al.* (1988) established similar results on moments of order statistics from some specific continuous distribution.

Order statistics play an important role in statistics in characterizing the distributions. The recurrence relations for single and product moments of order statistics obtained by Khan *et al.* (1983 a, b) have nicely been applied to characterize the probability distributions. Reference may be made to Govindarajulu (1975), Hwang and Lin (1984), Khan *et al.* (1987), Khan and Khan (1987), Khan and Ali (1987), Lin (1988, 1989), Huang (1989), Khan and Abu-Salih (1989), Kamps (1991) and others.

3. Distribution of single order statistic

Here in this section we will discuss the basic distribution theory of order statistics by assuming that population is absolutely continuous.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a continuous population having *pdf* $f(x)$ and *df* $F(x)$. Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ be the corresponding order statistics. The *pdf* of $X_{r:n}$, the r^{th} order statistics is given by (David and Nagaraja, 2003)

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}f(x)\,,$$

$$-\infty<x<\infty. \qquad (3.1)$$

The *pdf's* of smallest and largest order statistics are

$$f_{1:n}(x)=n\{1-F(x)\}^{n-1}f(x), \qquad -\infty<x<\infty \qquad (3.2)$$

and

$$f_{n:n}(x)=n\{F(x)\}^{n-1}f(x), \qquad -\infty<x<\infty \qquad (3.3)$$

The cumulative distribution functions of the smallest and the largest order statistics are easily derived by integrating the *pdf's* in (3.2) and (3.3) as

$$F_{1:n}(x)=1-\{1-F(x)\}^{n}, \qquad -\infty<x<\infty$$

and

$$F_{n:n}(x)=\{F(x)\}^{n}, \qquad -\infty<x<\infty.$$

In general, the *df* of $X_{r:n}$ is given by

$$F_{r:n}(x)=P(X_{r:n}\le x)$$

$$=P(\text{at least } r \text{ of } X_1,X_2,\cdots,X_n \text{ are less than or equal to } x)$$

$$=\sum_{i=r}^{n}P(\text{exactly } i \text{ of } X_1,X_2,\cdots,X_n \text{ are less than or equal to } x)$$

$$=\sum_{i=r}^{n}\binom{n}{i}\{F(x)\}^{i}\{1-F(x)\}^{n-i}\,, \quad -\infty<x<\infty \qquad (3.4)$$

The df of $X_{r:n}$ may also be obtained by integrating the pdf of $X_{r:n}$ in (3.1) as

$$F_{r:n}(x) = \int_{-\infty}^{x} f_{r:n}(t)dt$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{x} \{F(t)\}^{r-1}\{1-F(t)\}^{n-r} f(t)dt$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_{0}^{F(x)} u^{r-1}(1-u)^{n-r}\,du \tag{3.5}$$

$$= \mathrm{I}_{F(x)}(r, n-r+1), \tag{3.6}$$

which is Pearson's (1934) incomplete beta function. Equation (3.6) may also be obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991)

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{i+r-1}{r-1} \{F(x)\}^{r}\{1-F(x)\}^{i}, \quad -\infty < x < \infty \tag{3.7}$$

For continuous case the pdf of $X_{r:n}$ may also be obtained by differentiating (3.5) *w.r.t.* x.

The k^{th} $(k \geq 1)$ moment of $X_{r:n}$ is defined as

$$\mu_{r:n}^{(k)} = E(X_{r:n}^{k})$$

$$= \int_{-\infty}^{\infty} x^{k} f_{r:n}(x)\,dx \tag{3.8}$$

4. Joint distribution of two order statistics

The joint *pdf* of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s} f(x)f(y), \quad -\infty < x < y < \infty \tag{4.1}$$

The joint *df* of $X_{r:n}$ and $X_{s:n}$, $(1 \le r < s \le n)$ can be obtained as follows:

$$F_{r,s:n}(x,y) = P(X_{r:n} \le x, X_{s:n} \le y)$$

$= P($at least r of $X_1, X_2, \cdots, X_n$ are at most x and at

least s of $X_1, X_2, \cdots, X_n$ are at most $y)$.

$= \sum_{j=s}^{n} \sum_{i=r}^{j} P($exactly i of $X_1, X_2, \cdots, X_n$ are at most x

and exactly j of $X_1, X_2, \cdots, X_n$ are at most $y)$.

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!}\{F(x)\}^{i}$$

$$\times\{F(y)-F(x)\}^{j-i}\{1-F(y)\}^{n-j} \tag{4.2}$$

We can write the joint *df* of $X_{r:n}$ and $X_{s:n}$ in (4.2) equivalently as:

$$F_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_0^{F(x)}\int_0^{F(y)} u^{r-1}(v-u)^{s-r-1}$$

$$\times(1-v)^{n-s}\,du\,dv.$$

$$= I_{F(x),F(y)}(r, s-r, n-s+1), \quad -\infty < x < y < \infty, \tag{4.3}$$

which is incomplete bivariate beta function.

The product moments of the j^{th} and k^{th} order of $X_{r:n}$ and $X_{s:n}$ respectively, $(1 \le r < s \le n)$ is given by

$$\mu_{r,s:n}^{(j,k)} = E\left(X_{r:n}^{j}\, X_{s:n}^{k}\right) = \iint\limits_{-\infty<x<y<\infty} x^j y^k f_{r,s:n}(x,y)dxdy. \tag{4.4}$$

Remark 4.1: The ranking of random variables $X_1, X_2, \cdots, X_n$ is preserved under any monotonic increasing transformation of the random variables.

Remark 4.2: Regarding the probability integral transformation, if $X_{r:n}$, $1 \le r \le n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the r^{th} order statistic from a uniform distribution on $U(0,1)$.

Remark 4.3: Even if $X_1, X_2, \cdots, X_n$ are independent random variables, order statistics are not independent random variables.

Remark 4.4: Let $X_1, X_2, \cdots, X_n$ be *i.i.d.* random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, \cdots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

Remark 4.5: Let X be a continuous random variable with $E\left(X_{r:n}\right) = \alpha_{r:n}$,

i) If $\alpha = E(X)$ exits then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, $\alpha_{r:n}$ may exist for certain (but not all) values of r, even though α does not exist.

ii) $\alpha_{r:n}$ for all n determine the distribution completely.

5. Truncated and conditional distribution of order statistics

Let X be a continuous random variable having *pdf* $f(x)$ and *df* $F(x)$ in the interval $(-\infty,\infty)$.

Now if for the given P_1 and Q_1

Let

$$\int_{-\infty}^{Q_1} f(x)\,dx = Q \qquad \text{and} \qquad \int_{-\infty}^{P_1} f(x)\,dx = P, \tag{5.1}$$

This implies

$$\int_{Q_1}^{P_1} f(x)\,dx = P - Q \qquad \text{or,} \quad \frac{1}{P-Q}\int_{Q_1}^{P_1} f(x)\,dx = 1.$$

Thus doubly truncated *pdf* of X is given by

$$f_1(x) = \frac{f(x)}{P-Q}, \qquad x \in (Q_1, P_1) \tag{5.2}$$

and the corresponding *df* is

$$F_1(x) = \frac{1}{P-Q}\{F(x) - Q\}, \quad x \in (Q_1, P_1). \tag{5.3}$$

The lower and upper truncation points are Q_1, P_1 respectively; the proportions of truncation are Q on the left and $1-P$ on the right. If we put $Q=0$, the distribution will be truncated to the right. Similarly, for $P=1$, the distribution will be truncated to the left. Whereas for $Q=0$, $P=1$, we get the non-truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

In the following, we will relate the conditional distribution of order statistics (conditioned on another order statistic) to the distribution of order statistics from a population whose distribution is truncated from the original population distribution $F(x)$.

Statement 5.1: (David and Nagaraja, 2003): Let $X_1, X_2, \cdots, X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the r^{th} order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y.

Statement 5.2: (David and Nagaraja, 2003): Let $X_1, X_2, \cdots, X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$, given that $X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)^{th}$ order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x.

Statement 5.3: Let $X_1, X_2, \cdots, X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$ given that $X_{r:n} = x$ and $X_{t:n} = z$ for $1 \le r < s < t \le n$, is the same as the distribution of the $(s-r)^{th}$ order statistic obtained from a sample of size $(t-r-1)$ from a population whose distribution is truncated on the left at x and on the right at z.

Proof: The joint density functions of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$, $(1 \le r < s < t \le n)$ is given by

$$f_{r,s,t:n}(x,y,z) = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!}\{F(x)\}^{r-1}$$

$$\times\{F(y)-F(x)\}^{s-r-1}\{F(z)-F(y)\}^{t-s-1}$$

$$\times\{1-F(z)\}^{n-t} f(x) f(y) f(z),$$

$$-\infty < x < y < z < \infty \qquad (5.4)$$

From equation (5.4) we obtain the conditional density function of $X_{s:n}$ given that $X_{r:n} = x$ and $X_{t:n} = z$ to be

$$f_{s:n}(y \mid X_{r:n} = x, X_{t:n} = z) = \frac{f_{r,s,t:n}(x,y,z)}{f_{r,t:n}(x,z)}$$

$$= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)!}\left[\frac{F(y)-F(x)}{F(z)-F(x)}\right]^{s-r-1}$$

$$\times\left[1-\frac{F(y)-F(x)}{F(z)-F(x)}\right]^{t-s-1}\frac{f(y)}{F(z)-F(x)},$$

$$x < y < z. \qquad (5.5)$$

The result follows immediately from (5.5) upon noting that $\frac{F(y)-F(x)}{F(z)-F(x)}$ and $\frac{f(y)}{F(z)-F(x)}$ are the *df* and *pdf* of the population truncated on the left at x and on the right at z.

Remark 5.1: Statement 5.1 follows from Statement 5.3 by replacing t with $n+1$ with the convention $z = X_{n+1:n} = \beta$, where β is the upper range of X, $F(\beta)=1$.

Remark 5.2: Statement 5.2 follows from Statement 5.3 by letting $r=0$ with the convention $x = X_{0:n} = \alpha$ (lower limit).

Remark 5.3: Order statistics in a sample from a continuous distribution form a Markov chain, that is

$$f(X_{t:n} \mid X_{1:n} = x_1, \cdots, X_{r:n} = x_r, \cdots, X_{s:n} = x_s, \cdots, X_{n:n} = x_n)$$
$$= f(X_{t:n} \mid X_{r:n} = x_r,\, X_{s:n} = x_s).$$

So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

6. Some Continuous distributions

6.1: Pareto distribution

A random variable X is said to have the Pareto distribution if its *pdf* $f(x)$ and *df* $F(x)$ are of the form given below:

$$f(x) = v\,a^{v}\,x^{-(v+1)}, \qquad x \geq a, \qquad a, v > 0. \tag{6.1}$$

$$F(x) = 1 - a^{v} x^{-v}, \qquad x \geq a, \qquad a, v > 0. \tag{6.2}$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

6.2: Power function distribution

A random variable X is said to have a power function distribution if its *pdf* and *df* are of the form given below:

$$f(x) = v a^{-v} x^{v-1}, \qquad 0 \le x \le a, \qquad a, v > 0 \tag{6.3}$$

$$F(x) = a^{-v} x^{v}, \qquad 0 \le x \le a, \qquad a, v > 0. \tag{6.4}$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y = 1/X$ has a Pareto distribution.

6.3: Beta distribution

a) Beta distribution of first kind

A random variable X is said to have the beta distribution of first kind if its *pdf* is of the form

$$f(x) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}, \qquad 0 \le x \le 1, \ \ p, q > 0. \tag{6.5}$$

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r:n}$ is r^{th} order statistics from $U(0,1)$, then $X_{r:n}$ is distributed as $B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q = 1$, the distribution reduces to power function distribution.

b) Beta distribution of second kind

The *pdf* of beta distribution of second kind is given as

$$f(x) = \frac{1}{B(p,q)} \frac{x^{p-1}}{(1+x)^{p+q}}, \qquad p,q > 0, \ 0 \leq x < \infty \qquad (6.6)$$

is known as a beta variate of the second kind with parameters p and q.

Remark 6.1: Beta distribution of second kind reduces to beta distribution of first kind if we replace $1+x$ by $1/y$.

Usage: The Beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

6.4: Weibull distribution

A random variable X is said to have a Weibull distribution if its *pdf* is given by

$$f(x) = \theta\, p\, x^{p-1} e^{-\theta x^p}, \qquad 0 \leq x < \infty, \ \theta > 0, \ p > 0 \qquad (6.7)$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x^p}, \qquad 0 \leq x < \infty, \ \theta > 0, \ p > 0 \qquad (6.8)$$

Remark 6.2: If we put $p = 1$ in (6.7), we get the *pdf* of exponential distribution.

Remark 6.3: If we put $p = 2$, (6.7) gives *pdf* of Rayleigh distribution.

Remark 6.4: If X has a Weibull distribution, then the *pdf* of

$$Y = -p\log\left(\frac{X}{\alpha}\right) \text{ is}$$

$$f(y) = e^{-y}e^{-e^{-y}},$$

which is a form of an extreme value distribution.

Remark 6.5: The *pdf* and the *df* of inverse Weibull distribution are

$$f(x) = \theta\, p\, x^{-(p+1)}\, e^{-\theta x^{-p}}, \quad 0 \le x < \infty,\ \theta > 0,\ p > 0$$

$$F(x) = e^{-\theta\, x^{-p}}, \quad 0 \le x < \infty,\ \theta > 0,\ p > 0.$$

Usage: Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

6.5: Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f(x) = \theta\, e^{-\theta x}, \quad 0 \le x < \infty,\ \theta > 0 \tag{6.9}$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x}, \quad 0 \le x < \infty,\ \theta > 0. \tag{6.10}$$

Usage: The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption:

$$P(X > s+t \mid X > s) = P(X > t), \text{ for all } s \text{ and } t,$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

6.6: Rectangular distribution

A random variable X is said to have a rectangular distribution if its *pdf* is given by

$$f(x) = \frac{1}{\lambda - \beta}, \qquad \beta \le x \le \lambda$$

and the *df* is given by

$$F(x) = \frac{x-\beta}{\lambda - \beta}, \qquad \beta \le x \le \lambda.$$

The standard rectangular distribution $R(0,1)$ is obtained by putting $\beta = 0$ and $\lambda = 1$. It is noted that every distribution function $F(X)$ follows rectangular distribution $R(0,1)$. This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

6.7: Burr distribution

Let X be a continuous random variable, then different forms of *df* of X are listed below (Johnson and Kotz, 1970):

1 $\quad F(x) = x, \quad 0 < x < 1$

2 $\quad F(x) = (1+e^{-x})^{-k}, \quad -\infty < x < \infty$

3 $F(x) = (1+x^{-c})^{-k}, \quad 0 \le x < \infty$

4 $F(x) = \left[1+\left(\dfrac{c-x}{x}\right)^{1/c}\right]^{-k}, \; 0 \le k \le c$

5 $F(x) = [1+ce^{-\tan x}]^{-k}, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

6 $F(x) = [1+ce^{-k\sinh x}]^{-k}, \quad -\infty < x < \infty$

7 $F(x) = 2^{-k}(1+\tanh x)^{k}, \quad -\infty < x < \infty$

8 $F(x) = \left(\dfrac{2}{\pi}\tan^{-1} e^{x}\right), \quad -\infty < x < \infty$

9 $F(x) = 1 - \dfrac{2}{c[(1+e^{x})^{k}-1]+2}, \quad -\infty < x < \infty$

10 $F(x) = (1+e^{-x^2})^{k}, \quad 0 \le x < \infty$

11 $F(x) = \left(x - \dfrac{1}{2\pi}\sin 2\pi x\right)^{k}, \; 0 \le x \le 1$

12 $F(x) = 1-(1+x^{c})^{-k}, \; 0 \le x < \infty,$

where k and c are positive parameters.

Special attention is given to type XII, whose *pdf* is given as:

$$f(x) = kcx^{c-1}(1+x^{c})^{-(k+1)}, \quad 0 \le x < \infty, \; k, c > 0.$$

This is usually called Burr distribution. This distribution is frequently used for the purpose of graduation and in reliability theory. At $c=1$, it is called Lomax distribution whereas at $k=1$, it is known as Log-logistic distribution.

6.8: Log-logistic distribution

A random variable X follows the log-logistic if its *pdf* and *df* are

$$f(x)=\frac{p\theta x^{p-1}}{(1+\theta x^{p})^{2}}, \qquad 0\leq x<\infty \tag{6.11}$$

$$F(x)=1-\frac{1}{(1+\theta x^{p})^{-1}}, \qquad 0\leq x<\infty \tag{6.12}$$

6.9: Cauchy distribution

The special form of the Pearson type VII distribution, with *pdf*

$$f(x)=\frac{1}{\pi\lambda}\frac{1}{[1+\{(x-\theta)/\lambda\}^{2}]}, \qquad -\infty<x<\infty,$$

$$\lambda>0, \quad -\infty<\theta<\infty \tag{6.13}$$

is called the Cauchy distribution.

The *df* is given by

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right), \quad -\infty<x<\infty,$$

$$\lambda>0, \quad -\infty<\theta<\infty. \tag{6.14}$$

The distribution is symmetrical about $x=\theta$. The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting $\theta = 0$, $\lambda = 1$ and its *pdf* is given by

$$f(x) = \frac{1}{\pi} \frac{1}{(1+x^2)}, \quad -\infty < x < \infty \tag{6.15}$$

and the standard *df* is

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x, \quad -\infty < x < \infty.$$

Chapter II

MOMENTS OF ORDER STATISTICS

1. Introduction

In this chapter general results for finding the k^{th} moment of r^{th} order statistics are obtained without considering any particular distribution. These results are then utilized to obtain recurrence relations for doubly truncated and non-truncated distributions. The examples under consideration are Weibull, exponential, Pareto, power function, Cauchy, Burr, log-logistic, gamma and beta distributions. For applications of these distributions, one may refer to the Johnson and Kotz (1970).

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics obtained from a continuous *df* $F(x)$ and *pdf* $f(x)$. Then the *pdf* of $X_{r:n}$ $(1 \le r \le n)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x),$$

$$-\infty < x < \infty . \qquad (1.1)$$

Let $\alpha_{r:n}^{(k)} = E(X_{r:n}^{k})$, the k^{th} moment of r^{th} order statistic. Then

$$\alpha_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^k \{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x)dx. \qquad (1.2)$$

The *pdf* in case of truncation from both sides is

$$f(x)/(P-Q), \qquad Q_1 < x < P_1 \qquad (1.3)$$

where

$$\int_{-\infty}^{Q_1} f(x)\,dx = Q$$

and

$$\int_{P_1}^{\infty} f(x)\,dx = 1 - P \tag{1.4}$$

P and Q are assumed to be known ($Q<P$) and Q_1 and P_1 are functions of Q and P, respectively. For simplicity, $f(x)$ and $F(x)$ are used for truncation case as well, then in case of truncation from both sides:

$$\alpha_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} x^k \{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x)\,dx \tag{1.5}$$

and

$$\alpha_{1:n}^{(k)} = n\int_{Q_1}^{P_1} x^k \{1-F(x)\}^{n-1} f(x)\,dx. \tag{1.6}$$

2. Recurrence relations for moments of order statistics

Theorem 2.1: *(Khan et al., 1983 a)*

For Q_1 finite, $n \ge 1$ and $k = 1, 2, \ldots,$

$$\alpha_{1:n}^{(k)} = Q_1^k + k\int_{Q_1}^{P_1} x^{k-1}\{1-F(x)\}^n\,dx \tag{2.1}$$

Proof: Integrating (1.6) by parts, the above result is established.

Theorem 2.2: *(Khan et al., 1983 a)*

For $2 \le r \le n$, $n \ge 2$, and $k = 1, 2, \ldots,$

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} k\int_{Q_1}^{P_1} x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1}\,dx \tag{2.2}$$

under the assumptions

$$\lim_{x \to Q_1} x^k \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} = 0$$

and

$$\lim_{x \to P_1} x^k \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} = 0$$

Proof: Using (1.5) we have

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r} f(x) dx$$

$$- \frac{(n-1)!}{(r-2)!(n-r)!} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-2} \{1 - F(x)\}^{n-r} f(x) dx$$

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} \int_{Q_1}^{P_1} x^k \{F(x)\}^{r-2} \{1 - F(x)\}^{n-r}$$

$$\times \{nF(x) - (r-1)\} f(x) dx$$

Let

$$h(x) = -\{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} \tag{2.3}$$

Differentiating both sides of equation (2.3) with respect to x we get

$$h'(x) = -\{F(x)\}^{r-2} \{1 - F(x)\}^{n-r} \{nF(x) - (r-1)\} f(x) \tag{2.4}$$

Thus,

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} \int_{Q_1}^{P_1} x^k h'(x) dx . \tag{2.5}$$

Integrating (2.5) by parts and putting the value of $h(x)$ from (2.3) we get the result.

Also it can similarly be proved that

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n}^{(k)} = \binom{n}{r-1} k \int_{Q_1}^{P_1} x^{k-1} \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r+1} dx \tag{2.6}$$

and

$$\alpha_{r:n}^{(k)}-\alpha_{r:n-1}^{(k)}=-\binom{n-1}{r-1}k\int_{Q_1}^{P_1}x^{k-1}\{F(x)\}^r\{1-F(x)\}^{n-r}dx. \tag{2.7}$$

3. Recurrence relations for moments of order statistics for some specific distributions

Now we will use the results of Theorems 2.1 and 2.2 to obtain the recurrences relations for some specific distributions. It is assumed throughout that

$$\alpha_{r:n}^{(0)}=1, \qquad 1\le r\le n,$$

$$\alpha_{0:n}^{(k)}=Q_1^k, \qquad k=1,2,...,$$

and $\alpha_{n+1:n}^{(k)}=P_1^k \qquad k=1,2,...,$

In fact starting with $\alpha_{1:n}^{(k)}$, $k=1,2,...,$we can show that all the raw single moments of order statistics can be obtained systematically.

3.1: Doubly truncated Weibull and exponential distributions

The *pdf* of doubly truncated distribution is *(Khan et al., 1983 a)*

$$f(x)=\frac{px^{p-1}e^{-x^p}}{P-Q}, \qquad -\log(1-Q)\le x^p\le -\log(1-P), \qquad p>0.$$

Here $Q_1^p=-\log(1-Q)$ and $P_1^p=-\log(1-Q)$.

Let $Q_2=(1-Q)/(P-Q)$ and $P_2=(1-P)/(P-Q)$

then, $$\{1-F(x)\}=-P_2+\frac{1}{p}x^{1-p}f(x). \tag{3.1}$$

Putting the value of $\{1-F(x)\}$ from (3.1) in (2.1), we get

$$\alpha_{1:n}^{(k)} - Q_1^k = k\int_{Q_1}^{P_1} x^{k-1}\{1-F(x)\}^{n-1}\left\{-P_2 + \frac{1}{p}x^{1-p}f(x)\right\}dx$$

$$-P_2\left\{\alpha_{1:n-1}^{(k)} - Q_1^k\right\} + \frac{k}{np}n\int_{Q_1}^{P_1} x^{k-p}\{1-F(x)\}^{n-1}f(x)dx.$$

Thus in view of (2.1),

$$\alpha_{1:n}^{(k)} = Q_2 Q_1^k - P_2\,\alpha_{1:n-1}^{(k)} + \frac{k}{np}\alpha_{1:n}^{(k-p)} \tag{3.2}$$

so for $n = 1$,

$$\alpha_{1:1}^{(k)} = Q_2 Q_1^k - P_2 P_1^k + \frac{k}{p}\alpha_{1:1}^{(k-p)} \tag{3.3}$$

Now using Theorem 2.2, the relationship for the r^{th} order statistic is given by

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1}k\int_{Q_1}^{P_1} x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}$$

$$\times\left\{-P_2 + \frac{1}{p}x^{1-p}f(x)\right\}dx$$

$$= -P_2\frac{n-1}{n-r}\binom{n-2}{r-1}k\int_{Q_1}^{P_1} x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}dx$$

$$+\frac{k}{np}\frac{n!}{(r-1)!(n-r)!}$$

$$\times\int_{Q_1}^{P_1} x^{k-p}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}f(x)dx.$$

Therefore from (1.5) and (2.2), we get

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = -P_2\frac{n-1}{n-r}\left(\alpha_{r:n-1}^{(k)} - \alpha_{r-1:n-2}^{(k)}\right) + \frac{k}{np}\alpha_{r:n}^{(k-p)} \tag{3.4}$$

Using (1.5) and (1.6) we have the recurrence relation

$$r\alpha_{r+1:n}^{(k)} = n\alpha_{r:n-1}^{(k)} - (n-r)\alpha_{r:n}^{(k-p)} \ . \tag{3.5}$$

Rewriting (3.5) as

$$(r-1)\alpha_{r:n-1}^{(k)} = (n-1)\alpha_{r-1:n-2}^{(k)} - (n-r)\alpha_{r-1:n-1}^{(k-p)}$$

and using (3.4) we get

$$\alpha_{r:n}^{(k)} = Q_2\,\alpha_{r-1:n-1}^{(k)} - P_2\,\alpha_{r:n-1}^{(k)} + \frac{k}{np}\alpha_{r:n}^{(k-p)} \ . \tag{3.6}$$

For $r = n$, from (2.1) we get

$$\alpha_{n:n}^{(k)} = Q_2\,\alpha_{n-1:n-1}^{(k)} - P_2\,P_1^k + \frac{k}{np}\alpha_{n:n}^{(k-p)} \ .$$

Balakrishnan and Joshi (1981 a) had tried to obtain the recurrence relations between moments of order statistics for the non-truncated Weibull distribution, but they could not establish an explicit relationship. The method presented above is a generalization of the results of Balakrishnan and Joshi (1981 a). The exact and explicit expressions for the non-truncated Weibull distribution are given by Lieblein (1955). The results given by Joshi (1979 a) for doubly truncated exponential distributions are obtained by setting $p = 1$. Also the results by Joshi for non-truncated and right truncated exponential distribution are obtained respectively for $p = 1$, by setting $P = 1$ and $Q = 0$ in (3.2), (3.3), (3.5) and (3.6).

Saleh, Scott, and Junking (1975) have tabulated the first and second order moments of exponential distribution with $Q = 0$. Third and fourth order moments were tabulated by Joshi (1978).

3.2: Doubly truncated power function distribution

The *pdf* of doubly truncated power function distribution is *(Khan et al., 1983 a)*

$$f(x)=\frac{\nu a^{-\nu}x^{\nu-1}}{P-Q}, \qquad aQ^{1/\nu}\le x\le aP^{1/\nu} \qquad a,\nu>0.$$

Here $\quad Q_1 = aQ^{1/\nu}, \quad$ and $\quad P_1 = aP^{1/\nu}.$

Let $\quad Q_2 = Q/(P-Q). \quad$ and $\quad P_2 = P/(P-Q)$

then

$$\{1-F(x)\}=P_2-\frac{x}{\nu}f(x). \tag{3.7}$$

Thus, from (2.1) and (3.7),

$$\alpha_{1:n}^{(k)}-Q_1^k=P_2\left(\alpha_{1:n-1}^{(k)}-Q_1^k\right)-\frac{k}{n\nu}\alpha_{1:n}^{(k)}.$$

That is,

$$\alpha_{1:n}^{(k)}=\left(P_2\,\alpha_{1:n-1}^{(k)}-Q_2Q_1^k\right)\frac{n\nu}{n\nu+k}. \tag{3.8}$$

In view of (2.1), for $n=1$

$$\alpha_{1:1}^{(k)}=\left(P_2P_1^k-Q_2Q_1^k\right)\frac{\nu}{\nu+k}. \tag{3.9}$$

For r^{th} order statistic,

$$\alpha_{r:n}^{(k)}-\alpha_{r-1:n-1}^{(k)}=\frac{n-1}{n-r}P_2\left(\alpha_{r:n-1}^{(k)}-\alpha_{r-1:n-2}^{(k)}\right)-\frac{k}{n\nu}\alpha_{r:n}^{(k)}.$$

Using relation (3.6), we get

$$\alpha_{r:n}^{(k)}=\left(P_2\,\alpha_{r:n-1}^{(k)}-Q_2\alpha_{r-1:n-1}^{(k)}\right)\frac{n\nu}{n\nu+k} \tag{3.10}$$

Also for $r=n$ in view of (2.1), we get

$$\alpha_{n:n}^{(k)}=\left(P_2P_1^k-Q_2\alpha_{n-1:n-1}^{(k)}\right)\frac{n\nu}{n\nu+k}. \tag{3.11}$$

These relations were established by Balakrishnan and Joshi (1983). For the non-truncated case ($P=1, Q=0$), refer to Malik (1967).

3.3 Doubly truncated Pareto distributions

The *pdf* of doubly truncated Pareto distribution is *(Khan et al., 1983 a),*

$$f(x)=\frac{\nu a^{\nu} x^{-\nu-1}}{P-Q}, \qquad a(1-Q)^{-1/\nu} \le x \le a(1-P)^{-1/\nu}, \qquad a,\nu>0.$$

Here $\quad Q_1 = a(1-Q)^{-1/\nu} \quad$ and $\quad P_1 = a(1-P)^{-1/\nu}$

set $\quad Q_2 = (Q-1)/(P-Q) \quad$ and $\quad P_2 = (P-1)/(P-Q)$.

then,

$$\{1-F(x)\}=(x/\nu)f(x)+P_2.$$

Thus,

$$\alpha_{1:n}^{(k)} - Q_1^k = k\int_{Q_1}^{P_1} x^{k-1}\{1-F(x)\}^{n-1}\left\{\frac{x}{\nu}f(x)+P_2\right\}dx$$

$$=\frac{k}{n\nu}\alpha_{1:n}^{(k)} + P_2\left(\alpha_{1:n}^{(k)} - Q_1^k\right)$$

or

$$(n\nu-k)\alpha_{1:n}^{(k)} = \left(P_2\alpha_{1:n-1}^{(k)} + Q_2\, Q_1^k\right)n\nu, \qquad n\nu \ne k. \tag{3.12}$$

In particular,

$$(\nu-k)\alpha_{1:1}^{(k)} = \left(P_2\, P_1^k + Q_2\, Q_1^k\right)\nu, \qquad \nu \ne k. \tag{3.13}$$

For r^{th} order statistic, $2 \le r \le n-1$,

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \frac{k}{n\nu}\alpha_{r:n}^{(k)} + P_2\left(\frac{n-1}{n-r}\right)\left(\alpha_{r:n-1}^{(k)} - \alpha_{r-1:n-2}^{(k)}\right).$$

Using the recurrence relation given in (3.6) and simplifying, we get

$$(n\nu-k)\alpha_{r:n}^{(k)} = \left(P_2\, \alpha_{r:n-1}^{(k)} - Q_2\, \alpha_{r-1:n-1}^{(k)}\right)n\nu. \tag{3.14}$$

And for $r=n$ from (2.2), it can be seen that

$$(n\nu-k)\alpha_{n:n}^{(k)} = \left(P_2\, P_1^k - Q_2\, \alpha_{n-1:n-1}^{(k)}\right)n\nu, \quad n\nu \ne k. \tag{3.15}$$

In case $n\nu = k$, from (3.12), (3.14)and (3.15) we get, respectively,

$$\alpha_{1:n-1}^{(k)} = \frac{Q_2}{P_2} Q_1^k, \qquad n > 1 \tag{3.16}$$

$$\alpha_{r:n-1}^{(k)} = \frac{Q_2}{P_2} \alpha_{r-1:n-1}^{(k)}, \qquad 2 \le r < n-1 \tag{3.17}$$

$$\alpha_{n-1:n-1}^{(k)} = \frac{P_2}{Q_2} P_1^k. \tag{3.18}$$

However, this result may not be used to evaluate $\alpha_{1:1}^{(k)}$ and $\alpha_{n:n}^{(k)}$, when $n\nu = k$. For $\alpha_{1:1}^{(k)}$, it can easily be seen by direct integration that

$$\alpha_{1:1}^{(k)} = \frac{a^{\nu} \log(Q_2/P_2)}{P-Q}, \qquad \nu = k. \tag{3.19}$$

These results were established by Balakrishnan and Joshi (1982). For the non-truncated case refer to Malik (1966).

3.4: Doubly truncated Cauchy distribution

The *pdf* of doubly truncated Cauchy distribution is *(Khan et al., 1983 a)*

$$f(x) = \frac{1}{(P-Q)\pi} \frac{1}{1+x^2}, \qquad Q_1 \le x \le P_1,$$

where Q_1 and P_1 are obtained by

$$\int_{-\infty}^{Q_1} f(x)dx = Q,$$

$$\int_{P_1}^{\infty} f(x)dx = 1 - P.$$

Therefore,

$$(P-Q)\pi(1+x^2) f(x) = 1. \tag{3.20}$$

Now, in view of (2.2) and (3.20),

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)}$$

$$= \binom{n-1}{r-1} k \int_{Q_1}^{P_1} (P-Q)\pi(1+x^2)x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1} f(x)dx$$

$$= \frac{(n+1-r)k}{(n+1)n}\pi(P-Q)\left\{\alpha_{r:n+1}^{(k-1)} + \alpha_{r:n+1}^{(k+1)}\right\}$$

or

$$\alpha_{r:n+1}^{(k+1)} = \frac{n(n+1)}{\pi(P-Q)(n+1-r)k}\left\{\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)}\right\} - \alpha_{r:n+1}^{(k-1)}.$$

Replacing $n+1$ by n and $k+1$ by k, we get

$$\alpha_{r:n}^{(k)} = \frac{n(n-1)}{n(P-Q)(n-r)(k-1)}\left\{\alpha_{r:n-1}^{(k-1)} - \alpha_{r-1:n-2}^{(k-1)}\right\} - \alpha_{r:n}^{(k-2)}. \qquad (3.21)$$

Barnett (1966) has given the relation as

$$\alpha_{r:n}^{(k)} = \frac{n}{(k-1)\pi}\left\{\alpha_{r:n-1}^{(k-1)} - \alpha_{r-1:n-2}^{(k-1)}\right\} - \alpha_{r:n}^{(k-2)}. \qquad (3.22)$$

This could have been obtained from (2.6) and then replacing 1 by

$$(P-Q)\pi(1+x^2)f(x).$$

That is,

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n}^{(k)} = \binom{n}{r-1} k(P-Q)\pi$$

$$\times \int_{Q_1}^{P_1} (1+x^2)x^{k-1}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r+1} f(x)dx$$

$$= \frac{k\pi(P-Q)}{n+1}\left\{\alpha_{r:n+1}^{(k-1)} + \alpha_{r:n+1}^{(k+1)}\right\}. \tag{3.23}$$

Replacing $n+1$ by n and $k+1$ by k, and rearranging (3.23), we get (3.22) with $P=1$ and $Q=0$. Barnett (1966) has tabulated means of order statistics using the relation (3.22).

3.5: Doubly truncated Burr distribution

The *pdf* of doubly truncated Burr distribution is *(Khan and Khan, 1987),*

$$f(x) = \frac{mp\theta x^{p-1}(1+\theta x)^{-(m+1)}}{P-Q}, \qquad Q_1 \le x \le P_1.$$

Here $\quad \theta Q_1^p = \left\{(1-Q)^{-1/m} - 1\right\}$ and $\quad \theta P_1^p = \left\{(1-P)^{-1/m} - 1\right\}.$

Let $\quad Q_2 = \dfrac{1-Q}{P-Q}$ and $\quad P_2 = \dfrac{1-P}{P-Q}$

then,

$$\{1-F(x)\} = -P_2 + \frac{x^{1-p}}{mp\theta} f(x) + \frac{x}{mp} f(x).$$

Thus for $n > 1$, $k \ne mnp$,

$$\left[1 - \frac{k}{mnp}\right]\alpha_{1:n}^{(k)} = -P_2\alpha_{1:n-1}^{(k)} + Q_2 Q_1^k - \frac{k}{mnp\theta}\alpha_{1:n}^{(k-p)}. \tag{3.24}$$

For r^{th} order statistic, $2 \le r \le n-1$ and $k \ne mnp$

$$\left[1 - \frac{k}{mnp}\right]\alpha_{r:n}^{(k)} = Q_2\alpha_{r-1:n-1}^{(k)} - P_2\,\alpha_{r:n-1}^{(k)} + \frac{k}{mnp\theta}\alpha_{r:n}^{(k-p)}. \tag{3.25}$$

In particular, for $n=1$, $k \ne mnp$

$$\left[1 - \frac{k}{mp}\right]\alpha_{1:1}^{(k)} = -P_2 P_1^k + Q_2 Q_1^k - \frac{k}{mp\theta}\alpha_{1:1}^{(k-p)} \tag{3.26}$$

for $n > 1$

$$\left[1-\frac{k}{mnp}\right]\alpha_{n:n}^{(k)} = Q_2\alpha_{n-1:n-1}^{(k)} - P_2 P_1^k + \frac{k}{mnp\theta}\alpha_{n:n}^{(k-p)}$$

at $k = mnp$ and $2 \le r \le n-1$ we have from (3.25)

$$\alpha_{r:n-1}^{(k)} = \frac{Q_2}{P_2}\alpha_{r-1:n-1}^{(k)} + \frac{1}{\theta P_2}\alpha_{r:n}^{(k-p)}. \tag{3.27}$$

3.6: Doubly truncated log-logistic distribution

The *pdf* of doubly truncated logistic distribution is *(Al-Shboul and Khan, 1989)*,

$$f(x) = \frac{p\theta x^{p-1}}{(P-Q)(1+\theta x^p)^2}, \qquad Q_1 \le x \le P_1$$

Here $\quad \theta Q_1^p{}_1 = \dfrac{Q}{1-Q} \quad$ and $\quad \theta P_1^p = \dfrac{P}{1-P}$

or $\quad Q = \dfrac{\theta Q_1^p}{1+\theta Q_1^p} \quad$ and $\quad P = \dfrac{\theta P_1^p}{1+\theta P_1^p}$

we have

$$F(x) = \frac{1-Q}{P-Q} - \frac{1}{(P-Q)(1+\theta x^p)},$$

$$\{1-F(x)\} = \frac{1}{(P-Q)(1+\theta x^p)} - \frac{1-P}{P-Q}.$$

Thus, we have

$$F(x)\{1-F(x)\} = -\frac{P(1-P)}{(P-Q)^2} + \frac{(1-Q-P)}{(P-Q)}\{1-F(x)\}$$

$$+\frac{x}{p(P-Q)}f(x) \tag{3.28}$$

Thus in view of (2.2) and (3.28),

$$\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)} = \binom{n-1}{r-1} k \int_{Q_1}^{P_1} x^{k-1} \{F(x)\}^{r-2} \{1-F(x)\}^{n-r}$$

$$\times \left\{ -\frac{P(1-P)}{(P-Q)^2} + \frac{(1-Q-P)}{(P-Q)} \{1-F(x)\} + \frac{x}{p(P-Q)} f(x) \right\} dx$$

$$= -\frac{Q(1-Q)(n-1)}{(P-Q)^2 (r-1)} \left\{ \alpha_{r-1:n-2}^{(k)} - \alpha_{r-2:n-2}^{(k)} \right\}$$

$$+ \frac{(1-Q-P)(n-1)}{(P-Q)(r-1)} \left\{ \alpha_{r-1:n-1}^{(k)} - \alpha_{r-1:n-2}^{(k)} \right\}$$

$$+ \frac{k}{p(P-Q)(r-1)} \alpha_{r-1:n-1}^{(k)}.$$

That is,

$$\alpha_{r:n}^{(k)} = \left(1 + \frac{k}{p(P-Q)(r-1)} + \frac{(1-Q-P)(n-1)}{(P-Q)(r-1)} \right) \alpha_{r-1:n-1}^{(k)}$$

$$+ \left(\frac{P(1-P)(n-1)}{(P-Q)^2 (r-1)} \right) \alpha_{r-1:n-2}^{(k)}$$

$$+ \frac{Q(1-Q)(n-1)}{(P-Q)^2 (r-1)} \alpha_{r-2:n-2}^{(k)}. \tag{3.29}$$

If we set $P = 1$ and $Q = 0$ in (3.29), we get

$$\alpha_{r:n}^{(k)} = \left(1 + \frac{k}{p(r-1)} + \frac{(n-1)}{(r-1)} \right) \alpha_{r-1:n-1}^{(k)}. \tag{3.30}$$

3.7: Gamma distribution

The *pdf* of gamma distribution is *(Khan et al., 1983 a)*,

$$f(x) = \frac{e^{-x} x^{p-1}}{\Gamma(p)}, \qquad x, p > 0$$

$$\{1 - F(x)\} = \sum_{j=0}^{p-1} \frac{e^{-x} x^j}{j!}$$

$$= f(x)\Gamma(p)\sum_{j=0}^{p-1}\frac{x^{j+1-p}}{j!}. \tag{3.31}$$

Thus,

$$\alpha_{1:n}^{(k)} = \left(\frac{k}{n}\right)\Gamma(p)\sum_{j=0}^{p-1}\frac{\alpha_{1:n}^{(j+k-p)}}{j!}, \tag{3.32}$$

$$\alpha_{r:n}^{(k)} = \alpha_{r-1:n-1}^{(k)} + \left(\frac{k}{n}\right)\Gamma(p)\sum_{j=0}^{p-1}\frac{\alpha_{r:n}^{(j+k-p)}}{j!}. \tag{3.33}$$

These expressions were obtained by Joshi (1979). It may be noted that for $j+k<p$, some negative moments will also be involved. Joshi (1979) has tabulated the negative moments of these order statistics. Results of related interest may also be found in Gupta (1960) and Krishnaiah and Rizvi (1967). For $p=1$, (3.33) gives the recurrence relation for a non-truncated exponential distribution.

3.8: Beta distribution

The *pdf* of beta distribution is *(Khan et al., 1983 a),*

$$f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \qquad a,b>0 \qquad 0<x<1.$$

$$\{1-F(x)\}=\frac{1}{B(a,b)}\int_x^1 x^{a-1}(1-x)^{b-1}dx$$

$$=\sum_{j=0}^{a-1}\binom{a+b-1}{j}x^j(1-x)^{a+b-1-j}$$

$$= f(x)B(a,b)\sum_{j=0}^{a-1}\binom{a+b-1}{j}x^{j+1-a}(1-x)^{a-j}$$

$$= f(x)B(a,b)\sum_{j=0}^{a-1}\sum_{j=0}^{a-j}(-1)^l\binom{a+b-1}{j}\binom{a-j}{l}x^{j+l+1-a}\,. \tag{3.34}$$

Thus

$$\alpha_{r:n}^{(k)}-\alpha_{r-1:n-1}^{(k)}$$
$$=\frac{k}{n}B(a,b)\sum_{j=0}^{a-1}\sum_{j=0}^{a-j}(-1)^l\binom{a+b-1}{j}\binom{a-j}{l}\alpha_{r:n}^{(k+j+l-a+1)}\,. \tag{3.35}$$

This result is valid even for $r=1$ with $\alpha_{0:n}^{(k)}=0$.

PRODUCT MOMENTS OF ORDER STATISTICS

1. Introduction

In this chapter general results for obtaining the product moment of the j^{th} power of the r^{th} order statistic and the k^{th} power of the s^{th} order statistic are established. Then these results are utilized to determine recurrence relations for doubly truncated Weibull, exponential, Pareto, power function, Cauchy Burr and log-logistic distributions.

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics obtained from a continuous *df* $F(x)$ and *pdf* $f(x)$. Then the *pdf* of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ is given by

$$c_{r,s:n}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s} f(x)f(y),$$

$$-\infty < x < y < \infty. \tag{1.1}$$

where $$c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Let $$\alpha_{r,s:n}^{(j,k)} = E(X_{r:n}^{j}\ X_{s:n}^{k})$$

then

$$\alpha_{r,s:n}^{(j,k)} = c_{r,s:n} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^j y^k \{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s} f(x)f(y)\,dy\,dx. \tag{1.2}$$

The *pdf* in case of truncation from both sides is

$$\frac{f(x)}{P-Q}, \qquad Q_1 \le x \le P_1 \tag{1.3}$$

where

$$\int_{-\infty}^{Q_1} f(x)\,dx = Q \qquad \text{and} \qquad \int_{P_1}^{\infty} f(x)\,dx = 1-P.$$

P and Q are assumed to be known ($Q<P$) and Q_1 and P_1 are functions of Q and P, respectively. For simplicity, $f(x)$ and $F(x)$ are used for truncation cases as well, $\alpha_{r,s:n}^{(1,1)}$ will be denoted as $\alpha_{r,s:n}$. In case of truncation from both sides;

$$\alpha_{r,s:n}^{(j,k)} = c_{r,s:n} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^k \{F(x)\}^{r-1} \{F(y)-F(x)\}^{s-r-1}$$

$$\times \{1-F(y)\}^{n-s} f(x) f(y)\,dy\,dx. \tag{1.4}$$

2. Recurrence relations for product moments of order statistics

Theorem 2.1: *(Khan et al., 1983 b)*

For $(1 \le r < s \le n)$ and $j,k>0$,

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*}\, k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y)-F(x)\}^{s-r-1} \{1-F(y)\}^{n-s+1} f(x)\,dy\,dx \tag{2.1}$$

where $\quad c_{r,s-1:n}^{*} = \dfrac{n!}{(r-1)!(s-r-1)!(n-s+1)!} = \dfrac{c_{r,s-1:n}}{(s-r-1)}$

Proof: We have

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^k \{F(x)\}^{r-1}$$

$$\times\{F(y) - F(x)\}^{s-r-2}\{1 - F(y)\}^{n-s}$$

$$\times\{(n-r)F(y) - (n-s+1)F(x) - (s-r-1)\}$$

$$\times f(x) f(y)\, dy\, dx. \tag{2.2}$$

Let $$h(x,y) = -\{F(y) - F(x)\}^{s-r-1}\{1 - F(y)\}^{n-s+1}, \tag{2.3}$$

then $$\frac{\partial h(x,y)}{\partial y} = \{F(y) - F(x)\}^{s-r-2}\{1 - F(y)\}^{n-s}$$

$$\times\{(n-r)F(y) - (n-s+1)F(x) - (s-r-1)\} f(y). \tag{2.4}$$

Putting the value of (2.4) in (2.2), we get

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*} \int_{Q_1}^{P_1} x^j \{F(x)\}^{r-1} f(x)$$

$$\times\left\{\int_{x}^{P_1} y^k \frac{\partial}{\partial y} h(x,y)\, dy\right\} dx. \tag{2.5}$$

Now in view of (2.3),

$$\int_{x}^{P_1} y^k \frac{\partial}{\partial y} h(x,y)\, dy = k \int_{x}^{P_1} y^{k-1}\{F(y) - F(x)\}^{s-r-1}$$

$$\times\{1 - F(y)\}^{n-s+1}\, dy. \tag{2.6}$$

Substituting (2.6) in (2.5), the required expression is obtained.

Corollary 2.1: For $1 \le r \le n-1$ and $j, k > 0$,

$$\alpha_{r,r+1:n}^{(j,k)} = \alpha_{r:n}^{(j+k)} + c_{r:n}\, k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times\{1 - F(y)\}^{n-r} f(x)\, dy\, dx. \tag{2.7}$$

where

$$c_{r:n} = \frac{c_{r,r+1:n}}{(n-r)} = \frac{n!}{(r-1)!(n-r)!}$$

Proof: Putting $s = r+1$ in Theorem 2.1 and noting that

$$\alpha_{r,\,r:n}^{(j,k)} = E(X_{r:n}^{j}\, X_{r:n}^{k}) = E(X_{r:n}^{j+k}) = \alpha_{r:n}^{(j+k)}, \tag{2.8}$$

we get the desired result.

Corollary 2.2: For $n > 1$ and $j, k > 0$,

$$\alpha_{n-1,n:n}^{(j,k)} = \alpha_{n-1:n}^{(j,k)} + n(n-1)k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{n-2}$$

$$\times \{1 - F(y)\} f(x)\, dy\, dx. \tag{2.9}$$

Proof: Put $r = n-1$ in Theorem 2.1, to get the result.

Theorem 2.2: *(Khan et al., 1983 b)*

For $(1 \le r < s \le n)$ and $j > 0$,

$$\alpha_{r,s:n}^{(j,0)} = \alpha_{r,s-1:n}^{(j,0)} = \ldots \alpha_{r,r+1:n}^{(j,0)} = \alpha_{r:n}^{(j)}. \tag{2.10}$$

Proof: From relation (2.8), with $k = 0$

$$\alpha_{r,s:n}^{(j,0)} = E(X_{r:n}^{j}\, X_{s:n}^{0}) = E(X_{r:n}^{j}) = \alpha_{r:n}^{(j)}.$$

The usual technique in establishing the recurrence relations will be to express $\{1 - F(y)\}$ as a function of y and $f(y)$. It is interesting to note that the relations for specific distributions are obtained by just substitution.

3. Recurrence relations for product moments of order statistics for some specific distributions

3.1: Doubly truncated Weibull and exponential distributions

The *pdf* of the Weibull distribution is *(Khan et al., 1983 b),*

$$f(x)=\frac{px^{p-1}e^{-x^p}}{P-Q}, \qquad -\log(1-Q)\le x^p \le -\log(1-P), \quad p>0. \quad (3.1)$$

Here $\quad Q_1^p=-\log(1-Q) \quad$ and $\quad P_1^p=-\log(1-P)$.

Let $\quad Q_2=(1-Q)/(P-Q) \quad$ and $\quad P_2=(1-P)/(P-Q)$

then, $$\{1-F(y)\}=-P_2+\frac{1}{p}x^{1-p}f(y) \qquad (3.2)$$

Putting the value of $\{1-F(y)\}$ in (2.1), we get

$$\alpha_{r,s:n}^{(j,k)}-\alpha_{r,s-1:n}^{(j,k)}=c_{r,s-1:n}^{*}\,k\int_{Q_1}^{P_1}\int_{x}^{P_1}x^j y^{k-1}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s}\left\{-P_2+\frac{1}{p}x^{1-p}f(y)\right\}f(x)dy\,dx$$

$$=-P_2\frac{1}{n-s+1}c_{r,s:n}\,k\int_{Q_1}^{P_1}\int_{x}^{P_1}x^j y^{k-1}\{F(x)\}^{r-1}$$

$$\times\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s}f(x)dy\,dx$$

$$+\frac{k}{p(n-s+1)}c_{r,s:n}\int_{Q_1}^{P_1}\int_{x}^{P_1}x^j y^{k-p}\{F(x)\}^{r-1}$$

$$\times\{F(y)-F(x)\}^{s-r-1}\{1-F(y)\}^{n-s}f(x)f(y)dy\,dx.$$

That is,

$$\alpha_{r,s:n}^{(j,k)}=\alpha_{r,s-1:n}^{(j,k)}-\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}^{(j,k)}-\alpha_{r,s-1:n-1}^{(j,k)}\right)+\frac{k}{p(n-s+1)}\alpha_{r,s:n}^{(j,k-p)},$$

$$(1\le r<s\le n),\; s-r\ge 2. \qquad (3.3)$$

If we put $j=0$ in (3.3), we get

$$\alpha_{s:n}^{(k)}=\alpha_{s-1:n}^{(k)}-\frac{nP_2}{n-s+1}\left(\alpha_{s:n-1}^{(k)}-\alpha_{s-1:n-1}^{(k)}\right)+\frac{k}{p(n-s+1)}\alpha_{s:n}^{(k-p)}$$

a relation established for single moments given in Chapter II.

From Corollary 2.1, for $s=r+1$, (3.3) reduces to

$$\alpha_{r,r+1:n}^{(j,k)}=\alpha_{r:n}^{(j+k)}-\frac{nP_2}{(n-r)}\left(\alpha_{r,r+1:n-1}^{(j,k)}-\alpha_{r:n-1}^{(j+k)}\right)+\frac{k}{p(n-r)}\alpha_{r,r+1:n}^{(j,k-p)},$$

$$(1\le r\le n-1),\ n\ge 3. \tag{3.4}$$

From Corollary 2.2, for $r=n-1$ and $s=n$, (3.3) reduces to

$$\alpha_{n-1,n:n}^{(j,k)}=\alpha_{n-1:n}^{(j+k)}-nP_2\left(P_1^k\,\alpha_{n-1:n-1}^{(j)}-\alpha_{n-1:n-1}^{(j+k)}\right)+\frac{k}{p}\alpha_{n-1,n:n}^{(j,k-p)},$$

$$n\ge 2. \tag{3.5}$$

Expression (3.5) could also have been obtained from (3.4) by putting $r=n-1$. In substitution, we get a term $\alpha_{n-1,n:n-1}^{(j,k)}$ which is essentially an un defined term. This can be interpreted as $E(X_{n-1:n-1}^{j}P_1^k)=P_1^k\,\alpha_{n-1:n-1}^{(j)}$, where P_1 is the upper limit of the Weibull variate with the convention $X_{n:n-1}=P_1$.

If we put $p=1$ in the above expressions, we get corresponding results for the exponential distribution. For the non-truncated case one has to put $P=1$, $Q=0$. In case of doubly truncated Weibull distribution, recurrence relations for $\alpha_{r:n}^{(k)}$ are available in Khan *et al.* (1983 a). Expressions for exact and explicit product moment with $j=k=1$ can be obtained as in Leiblein (1955).

In case of $j=k=1$, for the Weibull distribution,

$$\alpha_{r,s:n}=\alpha_{r,s-1:n}-\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}-\alpha_{r,s-1:n-1}\right)+\frac{1}{p(n-s+1)}\alpha_{r,s:n}^{(1,1-p)},$$

$$(1\le r<s\le n),\ s-r\ge 2. \tag{3.6}$$

For the exponential distribution, this reduces to

$$\alpha_{r,s:n}=\alpha_{r,s-1:n}-\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}-\alpha_{r,s-1:n-1}\right)+\alpha_{r:n/(n-s+1)},$$

$$(1\le r<s\le n),\ s-r\ge 2. \tag{3.7}$$

in view of Theorem 2.2.

From (3.3), it is clear that if $k<p$, then we get negative moments, whose solution is given by Khan *et al.* (1984).

3.2: Doubly truncated power function distribution

The *pdf* of doubly truncated power function distribution is *(Khan et al., 1983 b),*

$$f(x)=\frac{va^{-v}x^{v-1}}{P-Q},\qquad aQ^{1/v}\le x\le aP^{1/v},\qquad a,v>0.$$

Here $\quad Q_1=aQ^{1/v}\quad$ and $\quad P_1=aP^{1/v}$.

Let $\quad Q_2=\dfrac{Q}{P-Q}\quad$ and $\quad P_2=\dfrac{P}{P-Q}$,

then, $$\{1-F(y)\}=P_2-\frac{y}{v}f(y). \tag{3.8}$$

Putting the value of $\{1-F(y)\}$ from (3.8) to Theorem 2.1, we get

$$\alpha_{r,s:n}^{(j,k)}-\alpha_{r,s-1:n}^{(j,k)}=c_{r,s-1:n}^{*}\,k\int_{Q_1}^{P_1}\int_{x}^{P_1}x^j y^{k-1}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s}\left\{P_2-\frac{y}{v}f(y)\right\}f(x)\,dy\,dx$$

$$=\frac{nP_2}{n-s+1}\left(\alpha_{r,s:n-1}^{(j,k)}-\alpha_{r,s-1:n-1}^{(j,k)}\right)-\frac{k}{v(n-s+1)}\alpha_{r,s:n}^{(j,k)}$$

On simplification, we get

$$\alpha_{r,s:n}^{(j,k)}=\frac{v}{v(n-s+1)+k}\left((n-s+1)\alpha_{r,s-1:n}^{(j,k)}+nP_2\left(\alpha_{r,s:n-1}^{(j,k)}-\alpha_{r,s-1:n-1}^{(j,k)}\right)\right),$$

$$(1\le r<s\le n),\ s-r\ge 2. \tag{3.9}$$

For $s=r+1$, we get from Corollary 2.1,

$$\alpha_{r,r+1:n}^{(j,k)}=\frac{v}{v(n-r)+k}\left((n-r)\alpha_{r:n}^{(j+k)}-nP_2\left(\alpha_{r,r+1:n-1}^{(j,k)}-\alpha_{r:n-1}^{(j+k)}\right)\right),$$

$$(1\le r\le n-2),\ n\ge 3 \tag{3.10}$$

after noting that $\alpha_{r,r:n}^{(j,k)}=\alpha_{r:n}^{(j+k)}$.

Similarly for $n=s=r+1$, we get

$$\alpha_{n-1,n:n}^{(j,k)}=\frac{v}{v+k}\left(\alpha_{n-1:n}^{(j+k)}+nP_2\left(P_1^k\,\alpha_{n-1:n-1}^{(j)}-\alpha_{n-1:n-1}^{(j+k)}\right)\right),$$

$$n\ge 2 \tag{3.11}$$

interpreting $\alpha_{n-1,n:n-1}^{(j,k)}=P_1^k\,\alpha_{n-1:n-1}^{(j)}$ as discussed in Section 3.1. Recurrence relations for single moments can be obtained by putting $j=0$ as given in Khan *et al.* (1983 a).

For $j=k=1$, the relations are available in Balakrishnan and Joshi (1981 b). The non-truncated cases ($P=1, Q=0$) are discussed by Malik (1967). Reference may also be made to Khan *et al.* (1983 a) for the recurrence relations of $\alpha_{r:n}^{(l)}$, $l=1,2,\ldots$.

3.3: Doubly truncated Pareto distribution

The *pdf* of doubly truncated Pareto distribution is *(Khan et al., 1983 b),*

$$f(x)=\frac{va^v x^{-v-1}}{P-Q},\quad a(1-Q)^{-1/v}\le x\le a(1-P)^{-1/v},\qquad a,v>0.$$

Here $\quad Q_1 = a(1-Q)^{-1/v} \quad$ and $\quad P_1 = a(1-P)^{-1/v}$

Let $\quad Q_2 = \dfrac{Q-1}{P-Q} \quad$ and $\quad P_2 = \dfrac{P-1}{P-Q}$,

then, $\quad \{1-F(y)\} = P_2 + \dfrac{y}{v} f(y)$. (3.12)

In view of (3.12) and (2.2),

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*} k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1} \{F(y)-F(x)\}^{s-r-1}$$

$$\times \{1-F(y)\}^{n-s} \left\{ P_2 + \frac{y}{v} f(y) \right\} f(x)\, dy\, dx$$

$$= \frac{nP_2}{n-s+1} \left(\alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)} \right)$$

$$+ \frac{k}{v(n-s+1)} \alpha_{r,s:n}^{(j,k)}.$$

Or

$$\left(v(n-s+1)-k\right)\alpha_{r,s:n}^{(j,k)} =$$

$$v\left((n-s+1)\alpha_{r,s-1:n}^{(j,k)} + nP_2\left(\alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)}\right)\right)$$

$$(1 \le r < s \le n),\ s-r \ge 2 \text{ and } v(n-s+1) \ne k. \qquad (3.13)$$

However, if $k = v\,(n-s+1)$, we get from (3.13),

$$(n-s+1)\,\alpha_{r,s-1:n}^{(j,k)} = nP_2\left(\alpha_{r,s-1:n-1}^{(j,k)} - \alpha_{r,s:n-1}^{(j,k)}\right). \qquad (3.14)$$

Marginal results for $k \ne v\,(n-s+1)$ can easily be seen to be equal to

$$(v(n-r)+k)\alpha_{r,r+1:n}^{(j,k)} =$$

$$v\left((n-r)\alpha_{r:n}^{(j+k)} + nP_2\left(\alpha_{r,r+1:n-1}^{(j,k)} - \alpha_{r:n-1}^{(j+k)}\right)\right),$$

$$(1 \le r \le n-2),\ n \ge 3. \qquad (3.15)$$

$$(v-k)\alpha_{n-1,n:n}^{(j,k)} = v\left(\alpha_{n-1:n}^{(j+k)} + nP_2\left(P_1^k\,\alpha_{n-1:n-1}^{(j)} - \alpha_{n-1:n-1}^{(j+k)}\right)\right),$$

$$n \geq 2. \qquad (3.16)$$

Recurrence relations for $j = k = 1$ have been obtained by Balakrishnan and Joshi (1982). Malik (1966) has obtained these results for $P = 1, Q = 0$. To evaluate $\alpha_{r,s:n}^{(j,k)}$, one may require the recurrence relations for $\alpha_{r:n}^{(l)}$ for which we refer to Khan *et al.* (1983 a) which can also be obtained by putting $j = 0$ and replacing s by r in this section.

3.4: Doubly truncated Cauchy distribution

The *pdf* of doubly truncated Cauchy distribution is *(Khan et al, 1983 b),*

$$f(x) = \frac{1}{(P-Q)\pi}\frac{1}{1+x^2}, \qquad Q_1 \leq x \leq P_1,$$

where Q_1 and P_1 are obtained as given in Section 1. Therefore,

$$1 = (P-Q)\pi(1+y^2)f(y). \qquad (3.17)$$

in view of (2.1) and (3.17),

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*}\,k\pi(P-Q)\int_{Q_1}^{P_1}\int_{x}^{P_1} x^j y^{k-1}(1+y^2)$$

$$\times\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}$$

$$\times\{1-F(y)\}^{n-s+1} f(x) f(y)\,dy\,dx$$

$$= \frac{k\pi(P-Q)}{n+1}\left(\alpha_{r,s:n+1}^{(j,k-1)} + \alpha_{r,s:n+1}^{(j,k+1)}\right).$$

Rearranging the terms, and replacing n by $(n-1)$, we get

$$\alpha_{r,s:n}^{(j,k+1)} = \frac{n}{\pi k(P-Q)}\left(\alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)}\right) - \alpha_{r,s:n}^{(j,k-1)},$$

$$1 \leq r < s \leq n-1. \qquad (3.18)$$

Similarly, it can be shown that

$$\alpha_{r,r+1:n}^{(j,k+1)} = \frac{n}{\pi k(P-Q)}\left(\alpha_{r,r+1:n-1}^{(j,k)} - \alpha_{r,:n-1}^{(j,k)}\right) - \alpha_{r,r+1:n}^{(j,k-1)},$$

$$1 \le r \le n-2 \qquad (3.19)$$

and

$$\alpha_{n-2,n-1:n}^{(j,k+1)} = \frac{n}{\pi k(P-Q)}\left(\alpha_{n-2,n-1:n-1}^{(j,k)} - \alpha_{n-2,:n-1}^{(j,k)}\right) - \alpha_{n-2,n-1:n}^{(j,k-1)},$$

$$n \ge 3. \qquad (3.20)$$

If we put $j = k = 1$, (3.18) reduces (in view of Theorem 2.2) to

$$\alpha_{r,s:n}^{(1,2)} = \frac{n}{\pi(P-Q)}\left(\alpha_{r,s:n-1} - \alpha_{r,s-1:n-1}\right) - \alpha_{r:n}. \qquad (3.21)$$

For the non-truncated case, put $P = 1, Q = 0$. For the recurrence relation of $\alpha_{r:n}^{(k)}$ in this case, one may refer to Barnett (1966). Reference may also be made to Khan *et al.* (1983 a).

3.5: Doubly truncated Burr distribution

The *pdf* of doubly truncated Burr distribution is *(Khan and Khan., 1987),*

$$f(x) = \frac{mp\theta x^{p-1}(1+\theta x)^{-(m+1)}}{P-Q}, \qquad Q_1 \le x \le P_1$$

where $\theta Q_1^p = \left((1-Q)^{-1/m} - 1\right)$ and $\theta P_1^p = \left((1-P)^{-1/m} - 1\right)$.

Let $Q_2 = \dfrac{1-Q}{P-Q}$ and $P_2 = \dfrac{1-P}{P-Q}$

Then

$$\{1 - F(y)\} = -P_2 + \frac{y^{1-p}}{mp\theta} f(y) + \frac{y}{mp} f(y). \qquad (3.22)$$

Putting the value of $\{1 - F(y)\}$ from (3.22) in Theorem 2.1, we get

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*} k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s+1}$$

$$\times \left\{ -P_2 + \frac{y^{1-p}}{mp\theta} f(y) + \frac{y}{mp} f(y) \right\} f(x)\, dy\, dx$$

$$= \frac{-nP_2}{(n-s+1)} \left\{ \alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)} \right\}$$

$$+ \frac{k}{(n-s+1)mp} \alpha_{r,s:n}^{(j,k)} + \frac{k}{(n-s+1)mp\theta} \alpha_{r,s:n}^{(j,k-p)}. \tag{3.23}$$

Rearranging the terms in the above equation we get

$$\left(1 - \frac{k}{(n-s+1)mp}\right) \alpha_{r,s:n}^{(j,k)} = -\frac{nP_2}{(n-s+1)} \left\{ \alpha_{r,s:n-1}^{(j,k)} - \alpha_{r,s-1:n-1}^{(j,k)} \right\}$$

$$+ \alpha_{r,s:n}^{(j,k)} + \frac{k}{(n-s+1)mp\theta} \alpha_{r,s:n}^{(j,k-p)},$$

$$k \neq (n-s+1)mp, \quad 1 \le r < s \le n-1, \quad s - r \ge 2. \tag{3.24}$$

Marginal results can be deduced from (3.24) by putting $s = r+1$ for $k \neq (n-r)mp, \quad 1 \le r \le n-2$.

$$\left(1 - \frac{k}{(n-r)mp}\right) \alpha_{r,r+1:n}^{(j,k)} = \alpha_{r:n}^{(j+k)} - \frac{nP_2}{(n-r)} \left\{ \alpha_{r,r+1:n-1}^{(j,k)} - \alpha_{r:n-1}^{(j+k)} \right\}$$

$$+ \frac{k}{(n-r)mp\theta} \alpha_{r,r+1:n}^{(j,k-p)}. \tag{3.25}$$

For $n = s = r+1$, $k \neq mp$ and $n \ge 2$

$$\left(1 - \frac{k}{mp}\right) \alpha_{n-1,n:n}^{(j,k)} = \alpha_{n-1:n}^{(j+k)} - nP_2 \left\{ P_1^k \alpha_{n-1:n-1}^{(j)} - \alpha_{n-1:n-1}^{(j+k)} \right\} + \frac{k}{mp\theta} \alpha_{n-1,n:n}^{(j,k-p)}.$$

3.6: Doubly truncated log-logistic distribution

The *pdf* of the doubly truncated log-logistic distribution is *(Al-Shboul and Khan, 1989),*

$$f(x)=\frac{p\theta x^{p-1}}{(P-Q)(1+\theta x^{p})^{2}},\qquad Q_1\le x\le P_1$$

Here $\theta Q_{1\,1}^{p}=\frac{Q}{1-Q}$ and $\theta P_1^{p}=\frac{P}{1-P}$

or $Q=\frac{\theta Q_1^{p}}{1+\theta Q_1^{p}}$ and $P=\frac{\theta P_1^{p}}{1+\theta P_1^{p}}$

we have

$$F(y)=\frac{1-Q}{P-Q}-\frac{1}{(P-Q)(1+\theta y^{p})},$$

$$\{1-F(y)\}=\frac{1}{(P-Q)(1+\theta y^{p})}-\frac{1-P}{P-Q}.$$

Thus, we have

$$F(y)\{1-F(y)\}=-\frac{Q(1-Q)}{(P-Q)^{2}}+\frac{(1-Q-P)}{(P-Q)}F(y)+\frac{y}{p(P-Q)}f(y)\qquad(3.26)$$

$$F(y)\{1-F(y)\}=-\frac{P(1-P)}{(P-Q)^{2}}+\frac{(1-Q-P)}{(P-Q)}\{1-F(y)\}+\frac{y}{p(P-Q)}f(y)\qquad(3.27)$$

$$\{1-F(y)\}^{2}=\frac{P(1-P)}{(P-Q)^{2}}+\frac{(2P-1)}{(P-Q)}\{1-F(y)\}-\frac{y}{p(P-Q)}f(y).\qquad(3.28)$$

Now from Theorem 2.2 and (3.28), we have

$$\alpha_{r,s:n}^{(j,k)} - \alpha_{r,s-1:n}^{(j,k)} = c_{r,s-1:n}^{*} k \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^{k-1} \{F(x)\}^{r-1}$$

$$\times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s-1}$$

$$\times \left\{ \frac{P(1-P)}{(P-Q)^2} + \frac{(2P-1)}{(P-Q)} \{1 - F(y)\} - \frac{y}{p(P-Q)} f(y) \right\}$$

$$\times f(x)\, dy\, dx$$

$$= \frac{n}{(n-s+1)} \left\{ \frac{2P-1}{P-Q} - \frac{k}{p(P-Q)(n-s)} \right\} \alpha_{r,s:n-1}^{(j,k)}$$

$$- \frac{n}{(n-s+1)} \frac{2P-1}{(P-Q)} \alpha_{r,s-1:n-1}^{(j,k)} + \frac{p(1-P)}{(P-Q)^2}$$

$$\times \frac{n(n-1)}{(n-s)(n-s+1)} \left\{ \alpha_{r,s:n-2}^{(j,k)} - \alpha_{r,s-1:n-2}^{(j,k)} \right\}$$

or, $$\alpha_{r,s:n}^{(j,k)} = \alpha_{r,s-1:n}^{(j,k)} + \frac{n}{(n-s+1)} \left\{ \frac{2P-1}{P-Q} - \frac{k}{p(P-Q)(n-s)} \right\} \alpha_{r,s:n-1}^{(j,k)}$$

$$- \frac{n}{(n-s+1)} \frac{2P-1}{(P-Q)} \alpha_{r,s-1:n-1}^{(j,k)} + \frac{p(1-P)}{(P-Q)^2}$$

$$\times \frac{n(n-1)}{(n-s)(n-s+1)} \left\{ \alpha_{r,s:n-2}^{(j,k)} - \alpha_{r,s-1:n-2}^{(j,k)} \right\}. \qquad (3.29)$$

Marginal results can be deduced from (3.29) by putting $s = r+1$, for $1 \le r \le n-2$

$$\alpha_{r,r+1:n}^{(j,k)} = \alpha_{r:n}^{(j+k)} + \frac{n}{(n-r)} \left\{ \frac{2P-1}{P-Q} - \frac{k}{p(P-Q)(n-r-1)} \right\} \alpha_{r,r+1:n-1}^{(j,k)}$$

$$- \frac{n}{(n-r)} \frac{2P-1}{(P-Q)^2} \alpha_{r:n-1}^{(j+k)} + \frac{p(1-P)}{(P-Q)^2}$$

$$\times \frac{n(n-1)}{(n-r-1)(n-r)} \left\{ \alpha_{r,r+1:n-2}^{(j,k)} - \alpha_{r:n-2}^{(j+k)} \right\}.$$

At $s=n$, we have

$$\alpha_{r,n:n}^{(j,k)}=\alpha_{r,n-1:n}^{(j,k)}+\frac{kn}{p(P-Q)(n-r-1)}\alpha_{r,n-1:n-1}^{(j,k)}+\frac{P(1-P)}{(P-Q)^2}$$

$$\times\frac{n(n-1)}{(n-r-1)}\alpha_{r,n-2:n-2}^{(j,k)}-\frac{P(1-P)}{(P-Q)^2}\frac{n(n-1)}{(n-r-1)}P_1^k\alpha_{r:n-2}^{(j)}$$

$$-\frac{r}{(n-r-1)}\left\{\alpha_{r+1,n:n}^{(j,k)}-\alpha_{r+1,n-1:n}^{(j,k)}\right\}+\frac{(1-Q-P)}{(P-Q)}$$

$$\times\frac{n}{(n-r-1)}\left\{\alpha_{r,n-1:n-1}^{(j,k)}-\alpha_{r,n-2:n-1}^{(j,k)}\right\}. \qquad (3.30)$$

Chapter IV

BOUNDS OF EXPECTATION FOR ORDER STATISTICS

1. Introduction

In this chapter we consider a number of general approaches giving bounds and approximations to the expectation of order statistics. Section 2 deals with the use of Cauchy-Schwarz inequality and some of its generalizations. Section 3 considers the case of bounds of expectation of r^{th} order statistics for unimodal and U-shaped distributions. In Section 4, sharp bounds are discussed and Section 5 presents an account on bounds via extremal dependences.

2. Bounds for expected value of order statistics in the *i.i.d.* case

Suppose $X_1, X_2, \ldots, X_n$ are *i.i.d* random variables with common distribution function $F(x)$. We assume that $F(x)$ has a finite mean μ and finite variance σ^2. Let $X_{1:n}, X_{2:n}, \ldots X_{n:n}$ be the order statistics and let $\mu_{r:n} = E(X_{r:n})$, $r = 1, 2, \ldots, n$. The *pdf* of $X_{n:n}$ is $n(F(x))^{n-1} f(x)$. Without loss of generality translate or rescale the X_i's so that $E(X) = 0$ and $E(X^2) = 1$.

Theorem 2.1: *(Gumbel, 1954)*

$$E(X_{n:n}) \leq (n-1)(2n-1)^{-1/2}$$

where equality holds when $F(x) = \left(\left(1 + \frac{x}{k}\right)\Big/ n\right)^{1/(n-1)}, \quad -k < x < \sqrt{2n-1}.$

Proof: Since

$$E(X_{n:n}) = n\int_{-\infty}^{\infty} x\{F(x)\}^{n-1} f(x)\,dx$$

we have,

$$E(X_{n:n}) = \int_0^1 F^{-1}(u)\,nu^{n-1}du \tag{2.1}$$

where $F(x)=u$. The Cauchy-Schwarz's inequality for square integrable functions g and h on (0,1) takes the form

$$\int_0^1 g(u)\,h(u)\,du \le \sqrt{\int_0^1 g^2(u)\,du \int_0^1 h^2(u)\,du} \tag{2.2}$$

with equality *if* and only *if* $g = kh$ *a.e.* on the sets where $g,\ h > 0$.

To apply the Cauchy-Schwarz's inequality, rewrite (2.1) in the form

$$E(X_{n:n}) = \int_0^1 F^{-1}(u)\,(nu^{n-1} - c)\,du. \tag{2.3}$$

Expression (2.3) is valid for every real c, since $\int_0^1 F^{-1}(u)\,du=0$, as $E(X)=0$.

To apply the Cauchy-Schwarz's inequality in (2.3), we identify

$$g(u)=F^{-1}(u)$$

and

$$h(u)=nu^{n-1} - c,$$

since $E(X^2) = \int_0^1 (F^{-1}(u))^2\,du=1$, this yields

$$E(X_{n:n}) \le \left(\int_0^1 (F^{-1}(u))^2\,du \int_0^1 (nu^{n-1} - c)^2\,du\right)^{1/2}$$

$$=\left(\int_0^1 (n^2u^{2(n-1)} - 2cnu^{n-1} + c^2)du\right)^{1/2}$$

$$=(c^2 - 2c + n^2(2n-1)^{-1})^{1/2}.$$

Let $A = c^2 - 2c + n^2(2n-1)^{-1}$.

Minimizing A with respect to c we find that A is minimum when $c = 1$. Setting $c = 1$, we have

$$E(X_{n:n}) \leq (n-1)(2n-1)^{-1/2}. \tag{2.4}$$

Equality is obtained in (2.4) if the common distribution function of the X_i's has an inverse which satisfies

$$F^{-1}(u) = k(nu^{n-1} - 1), \qquad 0 < u < 1 \tag{2.5}$$

the constant k in (2.5) must be chosen such that

$$E(X^2) = \int_0^1 (F^{-1}(u))^2 du = 1$$

or

$$k^2 \int_0^1 (n^2u^{2(n-1)} - 2nu^{n-1} + 1)du = 1$$

$$k^2(n^2(2n-1)^{-1} - 1) = 1$$

or

$$k = \sqrt{2n-1}/(n-1). \tag{2.6}$$

From (2.5)

$$F(k(nu^{n-1} - 1)) = u.$$

Let $\qquad x = k(nu^{n-1} - 1).$

then we find that extremal distribution is of the form

$$F(x)=\left(\left(1+\frac{x}{k}\right)\Big/n\right)^{1/(n-1)}, \qquad -k<x<\sqrt{2n-1} \tag{2.7}$$

where k is as given in (2.6). Here, if $n=2$ (2.7) reduces to a uniform distribution on the interval $(-\sqrt{3},\sqrt{3})$.

The extremal distribution given by (2.7) *(for n>2)* is rather unusual. In many situations, additional information about $F(x)$ might suggest that an extremal value for $E(X_{n:n})$ might be considerably less than the value provided by (2.4). For example, we might know that the common distribution of the X_i's is symmetric. This problem was actually treated by Moriguti (1951) before the general case was resolved.

Theorem 2.2: *(Moriguti, 1951)*

If the common distribution $F(x)$ of the X_i's is symmetric, then

$$E(X_{n:n})\le\frac{n}{\sqrt{2}}\sqrt{\frac{1}{2n-1}-B(n,n)}$$

where $$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

Proof: The requirement that $F(x)$ be symmetric can be written in terms of the inverse distribution function as follows

$$F^{-1}(u)=-F^{-1}(1-u), \qquad 0\le u\le 1. \tag{2.8}$$

If X is to have mean 0 and variance 1 then, in addition to (2.8), the only additional requirement is

$$\int_{1/2}^{1}(F^{-1}(u))^2\,du=\frac{1}{2}. \tag{2.9}$$

To determine the maximal value of $E(X_{n:n})$ for such a symmetric parent distribution we need to maximize (2.1) subject to (2.8) and (2.9). Equation (2.1) can be rewritten using (2.8) as

$$E(X_{n:n}) = \int_{1/2}^{1} F^{-1}(u)\, n(u^{n-1} - (1-u)^{n-1})\, du. \tag{2.10}$$

The Cauchy-Schwarz's inequality (2.2) may be applied to (2.10) using the choices

$$g(u) = F^{-1}(u), \qquad u > \frac{1}{2},$$

and

$$h(u) = n(u^{n-1} - (1-u)^{n-1}), \qquad u > \frac{1}{2}.$$

Using (2.9) this yields

$$\begin{aligned} E(X_{n:n}) &\le \left\{ \int_{1/2}^{1} (F^{-1}(u))^2\, du \int_{1/2}^{1} n^2 (u^{n-1} - (1-u)^{n-1})^2\, du \right\}^{1/2} \\ &= \sqrt{\frac{1}{2} \int_{1/2}^{1} n^2 (u^{n-1} - (1-u)^{n-1})^2\, du} \\ &= \frac{n}{\sqrt{2}} \sqrt{\frac{1}{2n-1} - B(n,n)}. \end{aligned} \tag{2.11}$$

The bound (2.11) is achievable by the distribution whose inverse is proportional to $n(u^{n-1} - (1-u)^{n-1})$ on the interval $\left(\frac{1}{2}, 1\right)$ and is extended to $\left(0, \frac{1}{2}\right)$ using (2.8). The required constant of proportionality is determined by the requirement that $Var(X) = 1$ (*i.e.* (2.9) must hold). Moriguti supplied graphs of the corresponding extremal densities for $n = 2,3,4,5,6,8,10$. It is interesting to observe that in both the cases $n = 2$ and $n = 3$, the extremal distribution is uniform on $(-\sqrt{3},\sqrt{3})$.

The extremal inverse distribution function (for which the bound (2.11) is achieved) is of the form

$$F^{-1}(u)=k(u^{n-1}-(1-u)^{n-1}), \qquad 0 \le u \le 1 \tag{2.12}$$

where

$$k=\frac{1}{\sqrt{2}}\left[\frac{1}{2n-1}-B(n,n)\right]^{-1/2}. \tag{2.13}$$

From (2.12), it is clear that the support of the extremal distribution function is $(-k,k)$ where k is given by (2.13). A closed form for the extremal distribution is not usually available (the exception being the cases $n = 2$ and $n = 3$, alluded to above).

In the later paper Moriguti (1953), considers bounds on the expectation of r^{th} order statistics.

Theorem 2.3: *(Balakrishnan, 1993)*

$$\left|E(X_{r:n})\right| \le \left[n\frac{\binom{2n-2r}{n-r}\binom{2r-2}{r-1}}{\binom{2n-1}{n-1}} - 1 \right]^{\frac{1}{2}}$$

Equality occurs if $F^{-1}(u) \propto b_{r:n}(u) - 1$, where $b_{r:n}(u)$ is the desity of the r^{th} order statistic from a uniform distribution.

Proof: Since

$$E(X_{r:n})=\int_0^1 F^{-1}(u)b_{r:n}(u)du. \tag{2.14}$$

Then, subject to the conditions

$$\int_0^1 F^{-1}(u)du=0 \qquad \text{and} \qquad \int_0^1 (F^{-1}(u))^2 du=1$$

We use Cauchy-Schwarz's inequality with the choices

$$g(u)=F^{-1}(u)$$

and

$$h(u)=b_{r:n}(u)-1$$

We have

$$\left|E(X_{n:n})\right| \le \left\{ \int_0^1 (F^{-1}(u))^2 du \int_0^1 (b_{r:n}(u)-1)^2 du \right\}^{1/2}$$

$$=\left[\int_0^1 \left(\frac{n!}{(r-1)!(n-r)!} u^{n-1}(1-u)^{n-1} \right)^2 du \right]^{1/2}$$

$$\le \left[n \frac{\binom{2n-2r}{n-r}\binom{2r-2}{r-1}}{\binom{2n-1}{n-1}} -1 \right]^{\frac{1}{2}}. \qquad (2.15)$$

Equality in (2.15) would occur if $F^{-1}(u) \propto b_{r:n}(u)-1$. Since $b_{r:n}(u)$ is only monotone for $r=1$ or $r=n$, the bound (2.15) is only sharp in these cases (F^{-1}being monotone cannot be proportional to a non-monotone function.). Moriguti suggested an ingenious way to determine a sharp bound by simply replacing $b_{r:n}$ by $h_{r:n}$, where, $h_{r:n}$ is an increasing density function chosen by considering all the distributions on $[0,1]$ corresponding to random variables which are stochastically larger than $U_{r:n}$ having increasing densities. Let $H_{r:n}$be the supremum of this class of densities and $h_{r:n}$ be its corresponding density. (In Moriguti's terms $H_{r:n}$ is the greatest convex minorant of $B_{r:n}$ $(=F_{U_{r:n}})$).

3. Bounds in case of unimodal and U-shaped distribution

$F(x)$ is unimodal(U-shaped) if there exist at least one real number c such that $g(u)$ is concave(convex) for $x<c$ and convex(concave) for $x>c$, where $x=g(u)$ is the inverse function of $F(x)$. This is because, for example, for any convex function $F(x)$ given $x_1<x_2$ and $0\le\alpha\le 1$.

$$\alpha F(x_1)+(1-\alpha)F(x_2)\ge F(\alpha x_1+(1-\alpha)x_2)$$

if and only *if*

$$g(\alpha F(x_1)+(1-\alpha)F(x_2))\ge\alpha x_1+(1-\alpha)x_2.$$

Theorem 3.1: *(Ali and Chan, 1965)*

Let $F(x)$ be symmetric, continuous and strictly increasing function. Then for $r\ge(n+1)/2$,

$$E(X_{r:n}) \ge g(r/n+1) \qquad \text{if } F(x) \text{ is unimodal.} \qquad (3.1)$$

where $x=F^{-1}(u)=g(u)$.

Proof: Without loss of generality, let $g\left(\frac{1}{2}\right)=0$ as the curve is symmetric over $(-\infty,\infty)$. For $r=(n+1)/2$, it is trivial that the equality in (3.1) holds. Hence we assume that $r>(n+1)/2$. Let

$$b_{r:n}(u)=n\binom{n-1}{r-1}u^{r-1}(1-u)^{n-r}. \qquad 0<u<1.$$

Then we have,

$$E(X_{r:n})=\int_0^1 g(u)b_{r:n}(u)\,du\,.$$

Also let $C=\int_{1/2}^{1}(b_{r:n}(u)-b_{r:n}(1-u))\,du$. By calculations it can be seen that

$(b_{r:n}(u)-b_{r:n}(1-u))>0$ for $\frac{1}{2}<u<1$ and that $0<C<1$.

The condition imposed on $F(x)$ imply that $g(u)$ is continuous and convex

for $\frac{1}{2}\leq u\leq 1$. Hence by Jensen's inequality, we have

$$E(X_{r:n})/C=\int_{1/2}^{1} g(u)\left\{(b_{r:n}(u)-b_{r:n}(1-u))/C\right\}du$$

$$\geq g\left(\int_{1/2}^{1} u\left\{(b_{r:n}(u)-b_{r:n}(1-u))/C\right\}du\right).$$

But

$$\int_{1/2}^{1} u\left\{(b_{r:n}(u)-b_{r:n}(1-u))/C\right\}du=\frac{1}{2}+(1/C)\left\{(r/(n+1))-\frac{1}{2}\right\}.$$

Then we have

$$E(X_{r:n})\geq C.g\left\{\frac{1}{2}+(1/C)\left((r/(n+1))-\frac{1}{2}\right)\right\}$$

$$=C.g\left\{\frac{1}{2}+(1/C)\left((r/(n+1))-\frac{1}{2}\right)\right\}+(1-C).\,g\left(\frac{1}{2}\right)$$

$$\geq g\,(r/(n+1))$$

since $g(u)$ is convex for $\frac{1}{2}\leq u\leq 1$ and $g\left(\frac{1}{2}\right)=0$.

Similar considerations show that for $r\geq(n+1)/2$

$$E(X_{r:n})\leq g(r/(n+1)) \qquad \text{if } F(x) \text{ is U-shaped.} \tag{3.2}$$

By symmetry we have, for $r<(n+1)/2$,

$$E(X_{r:n}) \le g(r/(n+1)) \qquad \text{if } F(x) \text{ is unimodal}$$

and

$$E(X_{r:n}) \ge g(r/(n+1)) \qquad \text{if } F(x) \text{ is U-shaped.}$$

Examples: The normal, the logistic, the Student's-t, the Laplace and the Cauchy distributions satisfy (3.1). For the distribution with probability density function

$$\{\Gamma(m+\tfrac{3}{2}) / (a\Gamma(\tfrac{1}{2})\ \Gamma(m+1))\}\,(1-(x^2/a^2))^m, \qquad -a \le x \le a$$

(3.1) is satisfied if $m \ge 0$ while (3.2) is satisfied when $-1 < m \le 0$. When $m = 0$, both (3.1) and (3.2) must be satisfied so that $E(X_{r:n}) = g(r/(n+1))$. Actually in this case the distribution is uniform distribution.

4. Sharp Bounds for the Expected value of order statistics

4.1: The upper bound (*Huang, 1997*)

The expected value of the r^{th} order statistics $X_{r:n}$ can be written as

$$E(X_{r:n}) = \int_0^1 F^{-1}(u)\, b_{r:n}(u)\,du$$

where $F^{-1}(u)$ is the quantile function, $F^{-1}(u) = \inf\{x : F(x) \ge u\}$.

$$b_{r:n}(u) = n\binom{n-1}{r-1} u^{r-1}(1-u)^{n-r}, \qquad 0<u<1.$$

Without loss of generality only the standardized $F(x)$ is considered, namely,

$$\int_0^1 F^{-1}(u)\,du = 0, \qquad \int_0^1 (F^{-1}(u))^2\,du = 1. \qquad (4.1)$$

Cauchy-Schwarz's inequality, subject to the constraint (4.1), yields the following upper bound:

$$E(X_{r:n}) = \int_0^1 F^{-1}(u)\left(b_{r:n}(u) - 1\right)du \le \left(\int_0^1 (b_{r:n}(u) - 1)^2 du\right)^{1/2}$$

$$= \left(n\binom{2n-1}{n-1}^{-1}\binom{2n-2r}{n-r}\binom{2r-2}{r-1} - 1\right)^{1/2}.$$

The bound is attainable only for the maximal order statistic $r = n$, namely,

$$E(X_{n:n}) \le \frac{n-1}{\sqrt{2n-1}}, \qquad n = 1, 2, \ldots$$

For $1 \le r < n$, a sharp upper bound is available through an elegant method of 'greatest convex minorant' due to Moriguti (1953).

Let $B_{r:n}$ be the distribution function corresponding to $b_{r:n}$; $\tilde{B}_{r:n}$ be the greatest convex minorant of $B_{r:n}$ and $\tilde{b}_{r:n}$ be its derivative. It follows that

$$E(X_{r:n}) = \int_0^1 F^{-1}(u)dB_{r:n}(u) \le \int_0^1 F^{-1}(u)d\tilde{B}_{r:n}(u)$$

$$= \int_0^1 F^{-1}(u)\tilde{b}_{r:n}(u)du = \int_0^1 F^{-1}(u)\left[\tilde{b}_{r:n}(u) - 1\right]du$$

$$\le \left(\int_0^1 \left(\tilde{b}_{r:n}(u) - 1\right)^2 du\right)^{1/2} = M_{r:n}$$

The upper bound $M_{r:n}$ is attainable by the distribution $F(x)$ whose inverse is proportional to $\tilde{b}_{r:n}(u) - 1$.

The difficulty here is in determining $\tilde{b}_{r:n}(u)$. It requires solving the polynomial equation of degree r–1,

$$(1-u)b_{r:n}(u) = 1 - B_{r:n}(u) \tag{4.2}$$

For $r = 2, 3$ the upper bound $M_{r:n}$ was given by Balakrishnan (1993), although, his $M_{3:n}$ was inaccurate. For $r > 4$ only numerical solutions are possible, and a table is given by Ludwig (1973).

4.2: The lower bound (*Huang, 1997*)

The lower bound requires no new calculation. It is clear that

$$E(X_{r:n}) \geq -M_{n-r+1:n} \tag{4.3}$$

Moreover, the bound is sharp, attainable by the negative of the distribution which attains the upper bound $M_{n-r+1:n}$. Thus sharp bounds

$$-M_{n-r+1:n} \leq E(X_{r:n}) \leq M_{r:n}$$

are asymmetric.

For illustration, taking $n = 4$, we have

$$-1.13389 \leq E(X_{1:4}) < 0, \qquad -0.50580 \leq E(X_{2:4}) \leq 0.18217,$$

$$-0.18217 \leq E(X_{3:4}) \leq 0.50580, \qquad 0 < E(X_{4:4}) \leq 1.13389.$$

It is obvious that the lower bound given by (4.3) was not pointed out by previous authors. In the literature, 'sharp' bounds were either given as $|E(X_{2:4})| \leq 0.18217$ or as $|E(X_{2:4})| \leq 0.50580$.

4.3: The symmetric case (*Huang, 1997*)

The same technique can be used to obtain sharp bounds for the symmetric $F(x)$. When $F(x)$ is symmetric, however, one of the bounds is trivial. Since $E(X_{r:n}) = 0$ for $r = \frac{1}{2}(n+1)$, it is clear that

$$E(X_{r:n}) < 0, \qquad r < \frac{n+1}{2}.$$

and

$$E(X_{r:n}) > 0, \qquad r > \frac{n+1}{2}. \tag{4.4}$$

Unlike the asymmetric case, these bounds are unattainable. For if $E(X_{r:n})=E(X_{s:n})$ for some $r<s$ then $F(x)$ degenerates, contradicting the assumption of $E(X^2)=1$. Nonetheless, these bounds are sharp, as demonstrated by the following example.

Consider the three point distribution $F(x)$, which puts mass $(2a^2)^{-1}$ at $a>1$ and at $-a$ the remaining mass at the origin. $F(x)$ is symmetric, with zero mean and unit variance. Simple calculation shows that

$$E(X_{n:n})=a\{1-(2a^2)^{-n}-(1-(2a^2)^{-1})^n\}\rightarrow 0$$

as $a\rightarrow\infty$. Thus, $E(X_{n:n})$ or $E(X_{r:n})$ can get arbitrarily close to zero, and the bound (4.4) cannot be improved.

5. Bounds on expectations of order statistics via extremal dependences

Caraux and Gascuel (1990, 1992) along with Rychlik (1990, 1992) established that, for identically distributed random variables

$$\sup\left(0,\ 1-\frac{n}{n-r+1}\{1-F(x)\}\right)\leq F_{X_{r:n}}(x)\leq \inf\left(\frac{n}{r}F(x),1\right) \tag{5.1}$$

and for non-identically distributed case

$$\sup\left(0,\ 1-\frac{\sum_{i=1}^{n}\{1-F_i(x)\}}{n-r+1}\right)\leq F_{X_{r:n}}(x)\leq \inf\left(\frac{\sum_{i=1}^{n}F_i(x)}{r},1\right) \tag{5.2}$$

5.1: Basic inequalities *(Gascuel and Caraux, 1992)*

For a given r, let $\bar{a}_r$ and $\underline{a}_r$ be the following quantiles

$$\bar{a}_r=F^{-1}\left(1-\frac{n-r-1}{n}\right) \quad \text{and} \quad \underline{a}_r=F^{-1}\left(\frac{r}{n}\right) \tag{5.3}$$

where

$$F^{-1}(u) = \inf(x \mid F(x) \geq u) \qquad \forall u \in [0,1]$$

then

$$\Pr(X_{r:n} \geq \bar{a}_r) = 1 \tag{5.4}$$

when variates X_i's are r-maximally dependent. Similarly when variates X_i's are r-minimally dependent, then

$$\Pr(X_{r:n} \leq \underline{a}_r) = 1.$$

Lai and Robbins (1976) established that when $E(X_1^+) < \infty$, then

$$E(X_{n:n}) \leq \bar{a}_n + n \int_{\bar{a}_n}^{\infty} \{1 - F(x)\} dx. \tag{5.5}$$

Proof: $E(X_{n:n})$ in the general case is smaller than or equal to $E(X_{n:n})$ for n-maximal dependence, in which case (5.5) is reached. This is obtained by integrating the survival function of the variate $(X_{n:n} - \bar{a}_n)$, which according to (5.4) and by using the lower bound (5.1), is certainly positive.

Rychlik (1990) provided a generalization of (5.5). When $-\infty < E(X_1^-)$ and $E(X_1^+) < \infty$, then

$$\underline{a}_r - \frac{n}{r} \int_{-\infty}^{\underline{a}_r} F(x) dx \leq E(X_{r:n}) \leq \bar{a}_r + \frac{n}{n-r+1} \int_{\bar{a}_r}^{\infty} \{1 - F(x)\} dx \tag{5.6}$$

The principle of the proof remains unchanged. The lower bound (upper bound) is reached when the variates are r-minimally (r-maximally) dependent.

Following Arnold (1980), (5.6) may be written by applying Fubini's theorem and convenient variable changes as,

$$\frac{n}{r} \int_0^{r/n} F^{-1}(u) du \leq E(X_{r:n}) \leq \frac{n}{n-r+1} \int_{1-(n-r+1)/n}^{1} F^{-1}(u) du. \tag{5.7}$$

Let us now consider the non-identically distributed case and bounds (5.2). We define $\overline{F}$ as the average of the F_i's,

$$\overline{F}=\frac{1}{n}\sum_i F_i$$

we obtain

$$\frac{n}{r}\int_0^{r/n}\overline{F}^{-1}(u)du\leq E(X_{r:n})\leq\frac{n}{n-r+1}\int_{1-(n-r+1)/n}^{1}\overline{F}^{-1}(u)du. \tag{5.8}$$

These bounds may be reached in some specific cases, e.g. for any *cdf* $F(x)$ when the variates are independently distributed.

5.2: Distribution free bounds

Let μ_i, σ_i and f_i be the expectation, variance and density of the X_i's respectively. We define $\overline{\mu}$ as the average of the μ_i's

$$\overline{\mu}=\frac{1}{n}\sum_i\mu_i .$$

Theorem 5.1: (*Gascuel and Caraux, 1992*)

The bounds (general bounds)

$$\overline{\mu}-\sqrt{\frac{(n-r)}{r}\frac{1}{n}\sum_{i=1}^{n}(\sigma_i^2+(\mu_i-\overline{\mu})^2)}\leq E(X_{r:n})\leq\overline{\mu}+\sqrt{\frac{(r-1)}{(n-r+1)}\frac{1}{n}\sum_{i=1}^{n}(\sigma_i^2+(\mu_i-\overline{\mu})^2)} \tag{5.9}$$

are valid and may be reached for some distributions.

Proof: Let us consider the lower bound, without loss of generality, let $\overline{\mu}=0$, i.e.

$$\overline{\mu}=\int_{-\infty}^{\infty}x\,d\overline{F}(x)=\int_0^1\overline{F}^{-1}(u)du=0.$$

Using the lower bound in (5.8), the Cauchy-Schwarz's inequality and the function $g=(n/r)1_{[0,r/n]}$, where $1_{[0,r/n]}$ is the indicator function of the set $[0,r/n]$, we have

$$
\begin{aligned}
E(X_{r:n}) &\geq \frac{n}{r}\int_0^{r/n} \overline{F}^{-1}(u)\,du \\
&= \int_0^1 (g(u)-1)\overline{F}^{-1}(u)\,du \\
&\geq -\left[\int_0^1 (g(u)-1)^2\,du \int_0^1 \left(\overline{F}^{-1}(u)\right)^2 du\right]^{1/2} \\
&= -\sqrt{\frac{n-r}{r}\frac{1}{n}\sum_{i=1}^{n}(\sigma_i^2+\mu_i^2)}\,.
\end{aligned}
$$

The bound for the general case $(\overline{\mu}\neq 0)$ is derived from the above expression. The lower-bound in (5.9) is reached (for $r\neq n$) when the X_i's are independently distributed *r*-minimally dependent and when they satisfy

$$
P\left(X_i=\mu-\sigma\sqrt{\frac{n-r}{r}}\right)=\frac{r}{n} \quad \text{and} \quad P\left(X_i=\mu+\sigma\sqrt{\frac{r}{n-r}}\right)=1-\frac{r}{n} \tag{5.10}
$$

where μ and σ^2 are the common expectation and variance of X_i's. Since the variates are *r*-minimally dependent, the lower bound in (5.8) is reached. Moreover, it is readily verified that with $\mu=0$, the inverse of the *cdf* corresponding to (5.10) is proportional to $g-1$. Therefore the Cauchy-Schwarz's inequality is achieved.

When $r=n$, it is sufficient to have $X_1=X_2=\cdots=X_n$.

The proof of the upper bound in (5.9) is similar.

Theorem 5.2: (*Gascuel and Caraux, 1992*)

The bounds (bounds with symmetrical distribution)

$$\left.\begin{matrix} \frac{r}{n}\le\frac{1}{2}, & \mu-\sigma\sqrt{\frac{n}{2r}} \\ \frac{r}{n}\ge\frac{1}{2}, & \mu-\sigma\sqrt{\frac{n(n-r)}{2r^2}} \end{matrix}\right\}\le E(X_{r:n})\le\begin{cases} \mu+\sigma\sqrt{\frac{n}{2(n-r+1)}} & ,\frac{r-1}{n}\ge\frac{1}{2} \\ \mu+\sigma\sqrt{\frac{n(r-1)}{2(n-r+1)^2}} & ,\frac{r-1}{n}\le\frac{1}{2} \end{cases} \tag{5.11}$$

are valid and may be reached for some distributions.

Proof: In the case of the lower-bound, for $\frac{r}{n}\le\frac{1}{2}$, and the case of the upper-bound, for $\frac{r-1}{n}\ge\frac{1}{2}$, the proofs are similar to those of given by Arnold (1980), they differ in the remaining cases.

For $\frac{r}{n}\le\frac{1}{2}$, without loss of generality, let $\mu=0$ and $\sigma=1$. Due to the symmetry of F^{-1} we have

$$\int_0^{r/n} F^{-1}(u)du=\int_{1-r/n}^{1} F^{-1}(u)du$$

and

$$\int_0^{1/2}(F^{-1}(u))^2\,du=\frac{1}{2}.$$

Let $g=(n/r)1_{[0,r/n]}$, using (5.7), the Cauchy-Schwarz's inequality and previous remarks, we obtain

$$E(X_{r:n})\ge-\left(\int_0^{1/2}(g(u))^2\,du\int_0^{1/2}(F^{-1}(u))^2\,du\right)^{1/2}=-\sqrt{\frac{n(n-r)}{2r^2}}.$$

The bound for the general case $(\mu \neq 0, \sigma \neq 1)$ is derived from this latter expression. The bound is reached for $r \neq n$, when the variates $X_1, X_2, \ldots, X_n$ are r-minimally dependent with common distribution of the following form:

$$P\left(X_i = \mu - \sigma\sqrt{\frac{n}{2(n-r)}}\right) = P\left(X_i = \mu + \sigma\sqrt{\frac{n}{2(n-r)}}\right) = 1 - \frac{r}{n} \quad (5.12a)$$

and

$$P(X_i = \mu) = \frac{2r}{n} - 1 \quad (5.12b)$$

It is readily verified that, with $\mu = 0$ and $\sigma = 1$, the inverse of the *cdf* $F(x)$ corresponding to (5.12 a, b) is proportional to g over the interval $[0, \frac{1}{2}]$.

When $r = n$, it is sufficient to have $X_1 = X_2 = \ldots = X_n$.

The proof of the upper bound when $\frac{r-1}{n} \leq \frac{1}{2}$ is similar.

5.3: Bounds for some usual distributions

Tighter bounds are naturally expected when the common distribution of the variates X_i's is known. The following tight bounds are easily derived from (5.6) or (5.7):

1. Bounds for the uniform distribution on $[0, 1]$:

$$\frac{1}{2}\frac{r}{n} \leq E(X_{r:n}) \leq 1 - \frac{1}{2}\frac{(n-r+1)}{n}$$

2. Bounds for the exponential distribution, when $F(x) = 1 - e^{-x}$ $(\forall\, x \geq 0)$:

$$1 + \frac{n-r}{r}\log\left(\frac{n-r}{r}\right) \leq E(X_{r:n}) \leq 1 + \log\left(\frac{n}{n-r+1}\right).$$

3. Bounds for the standard normal distribution: Let Φ and ϕ stand for *df* and *pdf* of standard normal distribution. Generalizing a result of Lai and Robbins (1976), Rychlik (1990) obtained the following bounds:

$$-\frac{n}{r}\phi\left(\Phi^{-1}\left(\frac{r}{n}\right)\right) \leq E(X_{r:n}) \leq \frac{n}{n-r+1}\phi\left(\Phi^{-1}\left(\frac{n-r+1}{n}\right)\right). \tag{5.13}$$

Theorem 5.3: (*Gascuel and Caraux, 1992*)

If the X_i's are independently distributed with standard normal distribution, then

$$-\sqrt{2\log\left(\frac{n}{r}\right)-\log\log\left(\frac{n}{r}\right)} \leq -\frac{n}{r}\phi\left(\Phi^{-1}\left(\frac{r}{n}\right)\right) \leq \mu_{r:n}, \qquad n/r \geq \sqrt{e}$$

and

$$E(X_{r:n}) \leq \frac{n}{n-r+1}\phi\left(\Phi^{-1}\left(\frac{n-r+1}{n}\right)\right)$$

$$\leq \sqrt{2\log\left(\frac{n}{n-r+1}\right)-\log\log\left(\frac{n}{n-r+1}\right)}, \qquad n/(n-r+1) \geq \sqrt{e}$$

Moreover, the lower bound (upper-bound) becomes sharp when $n/r \to \infty$ ($n/(n-r+1) \to \infty$).

Proof: These approximations are a generalization of the result obtained by Lai and Robbins (1976) for the maximum $(r=n)$ and the upper-bound of $E(X_{n:n})$, summarized as follows:

Let β be the function defined by $\beta(x) = x\phi(\Phi^{-1}(1/x))$ and let β^{+} be the function defined by $\beta^{+}(x) = \sqrt{2\log x - \log\log x}$. when $n \in \mathrm{N}$ and $\sqrt{e} \leq n \leq e^{7.5}$, Lai and Robbins (1976) numerically checked that

$\beta(n) \leq \beta^{+}(n)$. When $x \in R$ and $x \geq e^{7.5}$, they proved that $\beta(x) \leq \beta^{+}(x)$. For sharpness, they proved that $\lim\limits_{x \to \infty} (\beta^{+}(x) - \beta(x)) = 0$.

Let us deal with the upper-bound in Theorem 5.3. According to the above results, the only remaining point is to prove that when $n/(n-r+1) = x \notin \mathrm{N}$ and $\sqrt{e} \leq x \leq e^{7.5}$, then we still have $\beta(x) \leq \beta^{+}(x)$,as,

$$\beta \text{ and } \beta^{+} \text{ are increasing functions, when } x \geq \sqrt{e}. \tag{5.14}$$

Let i be the integer so that $i < x < i+1$. According to (5.14), to have $\beta(x) \leq \beta^{+}(x)$, it is sufficient to have $\beta(i+1) \leq \beta^{+}(i)$. This may be checked numerically for every integer i so that $\sqrt{e} \leq i \leq e^{7.5}$. Thus, the proof for the upper-bound, the lower-bound is completely symmetrical.

Chapter V

BOUNDS FOR VARIANCE, COVARIANCE AND CORRELATION OF ORDER STATISTICS

1. Introduction

In this chapter some recent developments in the field of bounds of variance, covariance and correlation of order statistics are discussed. In Section 2, an extension of Polya's inequality is discussed to derive bounds for the variance of maxima and minima of a random sample and then bounds for the variance of r^{th} order statistic. In Section 3, some light is thrown on covariance bounds between two order statistics and their functions. The conditions are also obtained when the equality is attained for samples of size two and three, thus leading to characterization of results. Some counterexamples to check the credibility of these bounds are also highlighted. In Section 4, bounds for correlation between the ordered sample pairs are discussed and the case that the correlation coefficient between the elements of order statistics is maximal when the underlying population is uniform is also discussed.

2. Bounds of variance of order statistics

We shall briefly report here some of the results on the upper bound of functions of random variables and order statistics.

2.1: The Polya inequality and the Abdelhamid inequality

Theorem 2.1: *(Hardy and Littlewood, 1932)*

If $g(x)$ is differentiable and square integrable over the unit interval then

$$\int_0^1 g^2(x)\,dx-\left[\int_0^1 g(x)\,dx\right]^2 \le \frac{1}{2}\int_0^1 x(1-x)(g'(x))^2\,dx \tag{2.1}$$

With equality if $g'(x)=c$ on $(0,1)$.

Theorem 2.2: *(Abdelhamid, 1985)*

Let X be an absolutely continuous random variable with distribution $F(x)$ and density $f(x)$ and finite second moment. Then

$$Var(X) \le \frac{1}{2}\int_0^1 u(1-u)(f(F^{-1}(u)))^{-2}du \tag{2.2}$$

with equality *if* and only *if* X has uniform distribution over (a,b) for some $a<b$.

Proof: Applying Theorem 2.1 with $g(x)=F^{-1}(x)$.

Remark: If X has mean 0, the change of variable $x=F^{-1}(u)$ in (2.2) yields

$$E(X^2) \le \frac{1}{2}\int_{-\infty}^{\infty} \frac{F(x)\{1-F(x)\}}{f^2(x)} f(x)\,dx. \tag{2.3}$$

A more striking equivalent statement is

$$E(X^2) \le \frac{1}{2}E\left(\frac{F(x)\{1-F(x)\}}{f^2(x)}\right). \tag{2.4}$$

Theorem 2.3: *(Abdelhamid, 1985)*

Let $X_{1:2}, X_{2:2}$ be the order statistics from an absolutely continuous distribution $F(x)$ with density $f(x)$ and finite variance. Then

$$Cov(X_{1:2}, X_{2:2}) \le \frac{1}{2}\int_0^1 u(1-u)(f(F^{-1}(u)))^{-2}du$$

$$-\frac{1}{2}(Var(X_{1:2})+Var(X_{2:2})) \tag{2.5}$$

with equality *if* and only *if* $f(x)$ is the uniform density on some interval (a,b).

Since,

$$Cov(X_{1:2},X_{2:2})+\frac{1}{2}(Var(X_{1:2})+Var(X_{2:2}))$$

$$=\frac{1}{2}Var(X_{1:2}+X_{2:2})$$

$$=\frac{1}{2}Var(X_1+X_2)$$

$$=Var(X_1)$$

or

$$Cov(X_{1:2},X_{2:2})=Var(X_1)-\frac{1}{2}(Var(X_{1:2})+Var(X_{2:2})).$$

Therefore, (2.5) follows in view of Theorem (2.2).

2.2: A minor extention of Polya's inequality

The Polya lemma can be interpreted as integrating $g(x)$ with respect to Lebesgue measure over $(0,1)$. Naturally a similar result can be obtained by considering alternatives measures. This idea rephrased in terms of random variables can be presented as follows.

Theorem 2.4: *(Arnold and Brockett, 1988)*

Suppose $Y=g(U)$ where the random variable $0<U<1$ has distribution function $F_U(u)$ and g is differentiable on $[0,1]$. It follows that, if *Var*(*Y*) exists, it satisfies

$$Var(Y)\leq E(U)\int_0^1(F_U(u)-F_U^{(1)}(u))(g'(u))^2\,du. \tag{2.6}$$

In which $F_U^{(1)}$ is the first moment distribution of U defined by

$$F_U^{(1)}(x)=\int_0^x u\,d\,F_U(u)/E(U) \tag{2.7}$$

Equality is attained *if* $g'(x)=c$ on $(0,1)$.

Proof: Let $U_{1:2}, U_{2:2}$ denote the order statistics from a sample of size 2 from F_U, then we have

$$Var(Y) = \frac{1}{2} E(g(U_{2:2}) - g(U_{1:2}))^2$$

$$= \int_0^1 d F_U(y) \int_0^y d F_U(x)(g(x) - g(y))^2$$

$$= \int_0^1 d F_U(y) \int_0^y d F_U(x)(y-x)^2 \left(\frac{1}{y-x} \int_x^y g'(u)\, du \right)^2$$

$$\le \int_0^1 d F_U(y) \int_0^y d F_U(x)(y-x)^2 \frac{1}{y-x} \int_x^y (g'(u))^2\, du$$

$$= \int_0^1 du \int_0^u d F_U(x) \int_u^1 d F_U(y)(y-x)(g'(u))^2$$

$$= E(U) \int_0^1 (F_U(u) - F_U^{(1)}(u))(g'(u))^2\, du .$$

It is clear that the above inequality becomes an equality if $g' = c$.

2.3: Variance bounds for maxima and minima

Theorem 2.5: *(Arnold and Brockett, 1988)*

Let $X_1, X_2, \ldots, X_n$ be an *i.i.d.* random sample with continuous *df* $F(x)$ and *pdf* $f(x)$. Then the variance of the largest order statistic

$$Var(X_{n:n}) \le \frac{n}{n+1} \int_0^1 u^n (1-u)(f(F^{-1}(u)))^{-2}\, du$$

where $\qquad x = F^{-1}(u), \qquad 0 < u < 1.$

Proof: For the largest order statistic we have

$$F_{X_{n:n}}(x) = F^n(x)$$

So that

$$F_{X_{n:n}}^{-1}(u) = F^{-1}(u^{1/n})$$

Since the density of $X_{n:n}$ is

$$f_{X_{n:n}}(x) = nF^{n-1}(x)f(x),$$

we have

$$f_{X_{n:n}}(F_{X_{n:n}}^{-1}(u)) = nu^{(n-1)/n} f(F^{-1}(u^{1/n})).$$

Also

$$Var(X) \le \frac{1}{2}\int_0^1 u(1-u)(f(F^{-1}(u)))^{-2}du \qquad \text{from (2.3).}$$

Substituting $f_{X_{n:n}}(F_{X_{n:n}}^{-1}(u))$ in the above inequality, we have

$$Var(X_{n:n}) \le \frac{1}{2}\int_0^1 u(1-u)\left(\frac{n.f(F^{-1}(u^{1/n}))}{u^{(1/n)-1}}\right)^{-2} du$$

$$= \frac{1}{2}\int_0^1 u(1-u)\left(\frac{\frac{1}{n}u^{(1/n)-1}}{f(F^{-1}(u^{1/n}))}\right)^{2} du.$$

Then a change of variable $t = u^{1/n}$ gives the bound

$$Var(X_{n:n}) \le \frac{1}{2}\int_0^1 \frac{t}{n}(1-t^n)(f(F^{-1}(t)))^{-2}dt. \tag{2.8}$$

Alternatively, $X_{n:n}$ has the same distribution as $F^{-1}(U_{n:n})$, where $U_{n:n}$ has distribution

$$F_{U_{n:n}}(u) = u^n, \qquad 0 < u < 1 \tag{2.9}$$

with

$$E(U_{n:n}) = n/(n+1)$$

and the first moment distribution as

$$F^{(1)}_{U_{n:n}}(u) = u^{n+1}, \qquad 0 < u < 1.$$

Using the above values the bound obtained by Theorem 2.4 is

$$Var(X_{n:n}) \le \frac{n}{n+1} \int_0^1 u^n (1-u)(f(F^{-1}(u)))^{-2} du. \tag{2.10}$$

The bounds provided by (2.8) and (2.10) can be trivial ($i.e. = +\infty$). There is no problem if $f(F^{-1}(u))$ is bounded away from zero, both bounds will be finite. A sufficient condition for the finiteness of both bounds is

$$\frac{1}{2} \int_0^1 u(1-u)(f(F^{-1}(u)))^{-2} du < \infty$$

(this is the common value of the bounds when $n = 1$).

Example 2.1: Consider the case where $F(x)$ is a power function distribution, i.e. $F(x) = x^{\alpha}$, $0 < x < 1$ where $\alpha > 0$. In this situation $F^{-1}(u) = u^{1/\alpha}$ and $f(x) = \alpha x^{\alpha-1}$

so that $\qquad f(F^{-1}(u)) = \alpha u^{1-(1/\alpha)}$

in this case (2.8) yields

$$Var(X_{n:n}) \le \frac{1}{2} \int_0^1 (1-u^n) \frac{u}{n} \alpha^{-2} u^{(2/\alpha)-2} du$$

$$= \frac{1}{4(2+\alpha n)} \tag{2.11}$$

while (2.10) gives

$$Var(X_{n:n}) \le \frac{n}{n+1} \int_0^1 u^n (1-u) \alpha^{-2} u^{(2/\alpha)-2} du$$

$$= \frac{n}{(2+\alpha n)(2+\alpha(n-1))(n+1)} \tag{2.12}$$

and the true variance of $X_{n:n}$ is

$$Var(X_{n:n}) = \frac{n\alpha}{(2+\alpha n)(n\alpha+1)^2}$$

It is readily verified that the bound (2.12) represents an improvement over the Abdelhamid bound (2.11) whenever $\alpha > 2(n+1)^{-1}$. Thus for any α, (2.12) will be better for sufficiently large n, the bound (2.11) behaves like $(4\alpha)^{-1}n^{-1}$while (2.12) is approximately $\alpha^{-2}n^{-2}$ and the true variance is approximately $\alpha^{-2}n^{-2}$. For large n, the bound provided by (2.12) is quite good.

An example in which $Var(X_{n:n})$ is difficult to compute while the bounds (2.8) and (2.10) are easily evaluated is:

Example 2.2: Let $h(u)=f(F^{-1}(u))$ assuming $F(x)$as strictly monotone, we have

$$\frac{1}{h(u)} = \frac{d}{du}F^{-1}(u).$$

So we may write

$$F^{-1}(u) = \int_{1/2}^{u} \frac{1}{h(t)}dt + c. \tag{2.13}$$

Now again let

$$h(u)=\sqrt{u(1-u)}$$

such that

$$h(u)=h(1-u)$$

so, the distribution is symmetric. Thus from (2.13) we have

$$F^{-1}(u)=\sin^{-1}(2u-1)+c'$$

whence

$$F(x)=(\sin(y-c')+1)/2, \qquad c'-\tfrac{\pi}{2}<y<c'+\tfrac{\pi}{2}.$$

This distribution, known as sine distribution, was studied by Burrows (1986). He showed that for samples from this distribution the maximum observation $X_{n:n}$ satisfies

$$E(X_{n:n})=c'+\frac{\pi}{2}\left(1-\binom{2n}{n}2^{-(2n-1)}\right).$$

Equation (2.8) yields

$$Var(X_{n:n})\le\frac{1}{2n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\frac{1}{2n}(\gamma+l^{nn})$$

where γ is Euler's constant, γ=0.5772..., while (2.10) gives a better bound

$$Var(X_{n:n})\le\frac{1}{(n+1)}.$$

For large values of n using Burrows' result one finds

$$Var(X_{n:n})=(4-\pi)n^{-1}.$$

Now by considering $X_{1:n}=-(X'_{n:n})$ where $X_i'=-X_i$, the versions of (2.8) and (2.10) relating to minima rather than maxima are:

$$Var(X_{1:n})\le\frac{1}{2}\int_0^1(1-(1-u)^n)\frac{(1-u)}{n}(f(F^{-1}(u)))^{-2}du \tag{2.14}$$

and $$Var(X_{1:n})\le\frac{n}{n+1}\int_0^1(1-u^n)u\,(f(F^{-1}(u)))^{-2}du. \tag{2.15}$$

Example 2.3: Let X_i's have Weibull distribution. Thus

$$F(x)=1-e^{-x^\gamma} \qquad x>0, \text{ for some } \gamma>0.$$

In this case we have

$$F^{-1}(u)=(-\log(1-u))^{1/\gamma} \text{ and } \quad f(x)=\gamma x^{\gamma-1}e^{-x^\gamma}, \qquad x>0.$$

Consequently,

$$f(F^{-1}(u)) = \gamma(1-u)(-\log(1-u))^{1-\gamma^{-1}}.$$

On substituting this in (2.14) the right hand side diverges to $+\infty$ for any n and any $\gamma>0$, so that (2.14) does not provide a non-trivial bound. Substitution in (2.15) however yields

$$Var(X_{1:n}) \leq \frac{n\gamma^{-2}}{n+1}\int_0^1 (1-u)^{n-2}u\,(-\log(1-u))^{2(\gamma^{-1}-1)}du$$

$$= \frac{n\gamma^{-2}}{n+1}\left((n-2)^{1-2\gamma^{-1}} - (n-1)^{1-2\gamma^{-1}}\right),$$

$$n>2,\ \gamma<2. \qquad (2.16)$$

In this case $X_{1:n}$ has a Weibull distribution and it is possible to write down the true variance

$$Var(X_{1:n}) = n^{-2\gamma^{-1}}\left(\Gamma(1+2\gamma^{-1}) + (\Gamma(1+\gamma^{-1}))^2\right). \qquad (2.17)$$

In the special case where $\gamma=1$, i.e. the exponential case, equation (2.16) provides the bound

$$Var(X_{1:n}) \leq \frac{n}{n-2}\frac{1}{(n^2-1)}$$

while the true variance from (2.17) is n^{-2}.

2.4: Bound of variance for the r^{th} order statistic

Cacoullos and Papathanasiou (1985) showed that

$$Var(g(x)) \leq \int_{-\infty}^{\infty} g'(x)^2 \int_{-\infty}^{x}(\mu-t)f(t)dt,$$

where equality holds *if* and only *if* $g(x)=ax+b$, $\mu=E(X)$, $f(x)$ the density of X. Substituting $h(u)=F^{-1}(u)=x$ (where U has the same distribution as in Arnold and Brockett's, (1988), Theorem 3.1) we have the following theorem.

Theorem 2.6: *(Papathanasiou, 1990)*

Let $X_1,\ldots,X_n$ be *i.i.d.* absolutely continuous *r.v's* from a population with *df* $F(x)$ (strictly monotone) and *pdf* $f(x)$. Then the following inequality holds:

$$Var(X_{r:n}) \leq \frac{n!}{(r-1)!(n-r)!(n+1)} \int_0^1 u^r (1-u)^{n-r+1} (h'(u))^2 \, du,$$

where equality is attained *if* and only *if* $h'(u)=c$ or *if* and only *if* $F(x)$ is a uniform distribution.

3. Bounds of covariance between two order statistics

Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables from a population with *df* $F(x)$ with finite variance, and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics. Bickel (1967) and Esary *et al.* (1967) showed that order statistics of independent random variables are non-negatively correlated, and for every pair of non-decreasing functions $g(.)$ and $h(.)$ with $Eg^2(X) < \infty$,

$$Cov\ (g(X_{r:n}), h(X_{s:n})) \geq 0. \tag{3.1}$$

Also, Papathanasiou (1990) proved that

$$Cov(X_{1:2}, X_{2:2}) \leq \tfrac{1}{3} Var(X) \tag{3.2}$$

where equality holds *if* and only *if* $F(x)$ is a uniform distribution.

3.1: Covariance of functions of order statistics

Theorem 3.1: (*Ma Chunsheng, 1992 a*)

Let $X_{1:2}, X_{2:2}$ be the order statistics from a sample of size 2 from an absolutely continuous *df* $F(x)$ with *pdf* $f(x)$, and $g(x)$ be any real-valued function with $Var(g(X)) < \infty$. Then

(i) $\quad Cov((g(X_{1:2}), g(X_{2:2})) = 4(Cov(g(X), F(X)))^2 \geq 0, \qquad (3.3)$

(ii) $\quad Cov\left(g\left(X_{1:2}\right), g\left(X_{2:2}\right)\right) \leq \frac{1}{3} Var\left(g(X)\right), \qquad (3.4)$

Equality holds *if* and only *if* $g(x) = c(F(x) - \frac{1}{2})$ for some constant c.

(iii) $$\frac{4}{3} Var(g(X)) \leq Var(g(X_{1:2})) + Var(g(X_{2:2}))$$
$$\leq 2Var(g(X)), \qquad (3.5)$$

where equality holds *if* and only *if* $Cov(g(X), F(X)) = 0$.

Proof: (i) We have

$$Eg(X_{1:2}).Eg(X_{2:2})$$
$$= \int_{-\infty}^{+\infty} g(x_1)\{2(1-F(x_1))f(x_1)\}dx_1 . \int_{-\infty}^{+\infty} g(x_2)\{2F(x_2)f(x_2)\}dx_2$$
$$= 4E\{g(X)(1-F(X))\}.E\{g(X)F(X)\}$$
$$= 4Eg(X).E\{g(X)F(X)\} - 4\{Eg(X)F(X)\}^2$$

and

$$E(g(X_{1:2})g(X_{2:2})) = \iint_{x_1 < x_2} g(x_1)g(x_2)\{2f(x_1)f(x_2)\}dx_1 dx_2$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x_1)g(x_2)f(x_1)f(x_2)dx_1\, dx_2$$
$$= (Eg(X))^2$$

Hence,

$$Cov(g(X_{1:2}), g(X_{2:2})) = 4(Eg(X)(F(X) - \tfrac{1}{2}))^2$$

$$= 4(Cov(g(X), F(X)))^2.$$

(ii) From (3.3) and the Cauchy-Schwartz inequality, we have

$$Cov(g(X_{1:2}), g(X_{2:2})) \leq 4Var(g(X)).Var(F(X)) = \tfrac{1}{3}Var(g(X))$$

where $Var(F(X)) = \int_{-\infty}^{+\infty}(F(x) - \tfrac{1}{2})^2 f(x)dx = \tfrac{1}{12}$.

(iii) It follows immediately that

$$E(g(X_{1:2}) + g(X_{2:2})) = 2E(g(X))$$

and

$$E(g^2(X_{1:2}) + g^2(X_{2:2})) = 2Eg^2(X).$$

so,

$$\begin{aligned} Var(g(X_{1:2}) + g(X_{2:2})) &= E(g(X_{1:2}) + g(X_{2:2}))^2 \\ &\quad - (E(g(X_{1:2}) + g(X_{2:2})))^2 \\ &= (2Eg^2(X) + 2Eg^2(X)) - (2Eg(X))^2 \\ &= 2Var(g(X)). \end{aligned}$$

On the other hand,

$$\begin{aligned} Var(g(X_{1:2}) + g(X_{2:2})) &= Var(g(X_{1:2})) + Var(g(X_{2:2})) \\ &\quad + 2Cov(g(X_{1:2}), g(X_{2:2})). \end{aligned}$$

Hence, we obtain (3.3) from (3.1) and (3.2).

Corollary 3.1: Under the conditions of the Theorem 3.1, $g(X_{1:2})$, $g(X_{2:2})$ are non-negatively correlated.

Corollary 3.2: $Cov(X_{1:2}, X_{2:2}) \leq \tfrac{1}{3}Var(X)$

where the equality holds *if* and only *if* $F(x)$ is a uniform distribution.

Theorem 3.2: (*Ma Chunsheng, 1992 b*)

For every $g \in G,\ h \in G$, the following identity holds

$$\sum_{r \neq s}\sum Cov(g(X_{r:n}), h(X_{s:n})))$$

$$= \sum_{r}(Eg(X_{r:n}).Eh(X_{r:n}) - Eg(X).Eh(X) \qquad (3.6)$$

where G denote the set of all real-valued functions $g(x)$ such that $Var(g(x)) < \infty$.

Proof: Clearly we have $\sum_{r} g(X_{r:n}) = \sum_{r} g(X_r)$ since the left-hand side is only a rearrangement of the right-hand side. Thus we have

$$E\sum_{r} g(X_{r:n}) = nEg(X),$$

and also

$$E\sum_{r} h(X_{r:n}) = nEh(X).$$

Moreover,

$$\sum_{r \neq s}\sum Eg(X_{r:n}).Eh(X_{s:n}) = \sum_{r}\left\{Eg(X_{r:n}).\left(\sum_{s} Eh(X_{s:n}) - Eh(X_{r:n})\right)\right\}$$

$$= n^2 Eg(X).Eh(X) - \sum_{r}\left\{Eg(X_{r:n}).Eh(X_{r:n})\right\}$$

(3.7)

Similarly, we have

$$\sum_{r \neq s}\sum E(g(X_{r:n})h(X_{s:n})) = E\left(\sum_{r \neq s}\sum g(X_{r:n})h(X_{s:n})\right)$$

$$= E\left(\sum_{r \neq s}\sum g(X_r)h(X_s)\right)$$

$$= \sum_{r \neq s}\sum E(g(X_r)h(X_s)) = \sum_{r \neq s}\sum Eg(X_r)Eh(X_s)$$

$$= n(n-1)Eg(X)Eh(X). \tag{3.8}$$

Finally, we obtain (3.6) from (3.7) and (3.8).

Corollary 3.3:

$$\sum_{r \neq s}\sum Cov(g(X_{r:n}), g(X_{s:n})) = \sum_{r}\{Eg(X_{r:n}) - Eg(X)\}^2 \geq 0 \tag{3.9}$$

where equality holds *if* and only *if* $Eg(X_{r:n}) = Eg(X)$ for $r = 1,2,\ldots,n$.

Corollary 3.4:

$$\frac{1}{n}\sum_{r} Var(g(X_{r:n})) \leq Var(g(X)). \tag{3.10}$$

Proof: Noting that

$$Var\left(\sum_{r} g(X_{r:n})\right) = Var\left(\sum_{r} g(X_r)\right) = nVar(g(X))$$

and

$$Var\left(\sum_{r} g(X_{r:n})\right) = \sum_{r} Var(g(X_{r:n})) + \sum_{r \neq s}\sum Cov(g(X_{r:n}), g(X_{r:n}))$$

we get (3.10) from (3.9).

Corollary 3.5:

$$\sum_{r \neq s}\sum E(g(X_{r:n})\ g(X_{s:n})) = n(n-1)\{E(g(X))\}^2 \geq 0.$$

Theorem 3.3: (*Ma Chunsheng, 1992 b*)

The following two statements are equivalent

(i) $Cov(X_{r:n}, X_{s:n}) = 0$ for some $r \neq s$. (3.11)

(ii) The distribution of X is degenerate.

Proof: (ii)$\Rightarrow$(i): Obvious.

(i)$\Rightarrow$(ii): By a result in Lehmann (1966), we have

$$Cov(X_{r:n}, X_{s:n}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(F_{(X_{r:n}, X_{s:n})}(x, y) - F_{X_{r:n}}(x).F_{X_{s:n}}(y)\right)dxdy \tag{3.12}$$

Esary *et al.* (1967) proved that $X_{r:n}$ and $X_{s:n}$ $(r \neq s)$ are positively quadrant dependent, *i.e.*, for all x, y.

$$P(X_{r:n} \leq x, X_{s:n} \leq y) \geq P(X_{r:n} \leq x).P(X_{s:n} \leq y)$$

or

$$F_{(X_{r:n}, X_{s:n})}(x, y) \geq F_{X_{r:n}}(x).F_{X_{s:n}}(y).$$

That is, the integrand in (3.12) is non-negative. Hence, by (3.11) this integrand must be zero, i.e.,

$$F_{(X_{r:n}, X_{s:n})}(x, y) = F_{X_{r:n}}(x).F_{X_{s:n}}(y).$$

Thus $X_{r:n}$ and $X_{s:n}$ are independent.

To prove that X is degenerate, we need only consider the case $r=1$ and $s=n$. From the independence between $X_{1:n}$ and, $X_{n:n}$ we have

$$P(X_{1:n} \geq x, X_{n:n} \leq y) = P(X_{1:n} \geq x).P(X_{n:n} \leq y) \text{ for } x \leq y.$$

But

$$P(X_{1:n} \geq x, X_{n:n} \leq y) = \prod_{r} P(x \leq X_r \leq y) = \{F(y) - F(x)\}^n$$

and

$$P(X_{1:n} \geq x) = \{1 - F(x)\}^n, \qquad P(X_{n:n} \leq y) = (F(y))^n.$$

Thus we obtain

$$F(y) - F(x) = \{1 - F(x)\}.F(y)$$

or

$$F(x).\{1 - F(y)\} = 0.$$

Hence, the distribution of X is degenerate.

Corollary 3.6: The equality in (3.1) holds *if* and only *if* one of the two statements holds

(i) Either $g(.)$ or $h(.)$ is constant, *a.e.*

(ii) The distribution of X is degenerate.

Theorem 3.4: (*Ma Chunsheng, 1992 b*)

Suppose $F(x)$ is continuous. Then

(i) $\sum\sum_{r \neq s} Cov\ (g(X_{r:n}), h(X_{s:n}))$

$$= \sum_r \left\{ \begin{aligned} &\left[Cov\left(g(X), \frac{1}{B(r,n-r+1)} \{F(X)\}^{r-1}\{1-F(X)\}^{n-r} \right)\right] \\ &.\left[Cov\left(h(X), \frac{1}{B(r,n-r+1)} \{F(X)\}^{r-1}\{1-F(X)\}^{n-r} \right)\right] \end{aligned} \right\}. \tag{3.13}$$

(ii) $\sum\sum_{r \neq s} Cov\ (g(X_{r:n}), h(X_{s:n}))$

$$= \sum_r \left[Cov\left(g(X), \frac{1}{B(r,n-r+1)} \{F(X)\}^{r-1}\{1-F(X)\}^{n-r} \right)\right]^2$$

$$\leq \left[\sum_r \left(\frac{B(2r-1,2n-2r-1)}{[B(r,n-r+1)]^2} - 1 \right)\right] .Var(g(X)). \tag{3.14}$$

(iii) $\displaystyle \frac{1}{n}\sum_r Var(g(X_{r:n})) \geq \sum_r \left(\frac{B(2r-1,2n-2r-1)}{[B(r,n-r+1)]^2} \right) Var(g(X)).$

Proof: (i) It follows immediately that

$$Eg(X_{r:n}) = \frac{1}{B(r,n-r+1)} E\left\{ g(X)\{F(X)\}^{r-1}\{1-F(X)\}^{n-r} \right\}$$

and

$$E\{\{F(X)\}^{r-1}\{1-F(X)\}^{n-r}\}=B(r,n-r+1).$$

Thus

$$Cov\left(g(X),\frac{1}{B(n-r+1)}\{\{F(X)\}^{r-1}\{1-F(X)\}^{n-r}\}\right)$$

$$=Eg(X_{r:n})-Eg(X).$$

Similarly, we have

$$Cov\left(h(X),\frac{1}{B(n-r+1)}\{\{F(X)\}^{r-1}\{1-F(X)\}^{n-r}\}\right)$$

$$=Eh(X_{r:n})-Eh(X).$$

After a simple calculation, we get (3.13).

(ii) Applying the Cauchy-Schwartz inequality, we obtain

$$\left[Cov\left(g(X),\frac{1}{B(r,n-r+1)}\{F(X)\}^{r-1}\{1-F(X)\}^{n-r}\right)\right]^2$$

$$\leq\left(\frac{B(2r-1,2n-2r-1)}{[B(r,n-r+1)]^2}-1\right)Var(g(X))$$

and thus the desired result (3.14).

(iii) It follows from the proof of corollary 3.4.

Corollary 3.7:

$$\lim_{n\to+\infty}\left[\frac{1}{n}\sum_r Var(g(X_{r:n}))\right]=Var(g(X)).$$

Corollary 3.8: For $n = 2$ and $g\in G$, $h\in G$, we have:

(i) $Cov(g(X_{1:2}),h(X_{2:2}))+Cov(g(X_{2:2}),h(X_{1:2}))$

$$=8\,Cov(g(X),F(X)).Cov(h(X),F(X)). \qquad (3.15)$$

(ii) $0\leq Cov(g(X_{1:2}),g(X_{2:2}))$

$$=4\{Cov(g(X),F(X))\}^2\leq\tfrac{1}{3}Var(g(X))$$

where G denote the set of all real-valued functions $g(x)$ such that $Var(g(x)) < \infty$.

Theorem 3.5: (*Ma Chunsheng, 1992 b*)

Suppose that $F(x)$ is continuous then

$$0 < \sup_{g,h}\{Cov(g(X_{1:2}),h(X_{2:2}))\} \le \tfrac{1}{3}\sup_{g}\{Var(g(X))\} \tag{3.16}$$

where supremum is taken for any non-decreasing function with finite variance.

Proof: If $\sup_{g}\{Var(g(X))\} = \infty$, (3.16) is obvious.

Now suppose that $\sup_{g,h}\{Cov(g(X_{1:2}),h(X_{2:2}))\} < \infty$. Take g or h = F in (3.15), to obtain $Cov(g(X),F(X)) \ge 0$, $Cov(h(X),F(X)) \ge 0$. Then from (3.1) and (3.15), we have

$$\sup_{g,h}\{Cov(g(X_{1:2}),h(X_{2:2}))\}$$

$$= \frac{1}{2}\sup_{g,h}\{Cov(g(X_{1:2}),h(X_{2:2})) + Cov(g(X_{2:2}),h(X_{1:2}))\}$$

$$= 4\sup_{g}\{Cov(g(X),F(X)).Cov(h(X),F(X))\}$$

$$= 4\sup_{g}\{Cov(g(X),F(X))\}^2$$

$$\le \tfrac{1}{3}Var(g(X)) \qquad \text{(by Cauchy-Schwartz's inequality).}$$

Corollary 3.9: Let G_1 (G_2) denote the set of all non-decreasing (non-increasing) functions $g(x)$, with $Var(g(X)) = 1$. Then

(i) $\max_{g,h}\{Cov(g(X_{1:2}),h(X_{2:2}))\} = \dfrac{1}{3}$ for all $g,h \in G_1$

where the maximum can be achieved when $g = h = 2\sqrt{3}F$.

(ii) $\max_{g,h} \{Cov\,(g(X_{1:2}), h(X_{2:2}))\} = \frac{1}{3}$ for all $g, h \in G_2$

where the maximum can be achieved when $g = h = -2\sqrt{3}F$.

Theorem 3.6: (*Yong-cheng Qi, 1994*)

Suppose $n \geq 2$ and $1 \leq r \leq n-1$ and g is a real-valued function with $Eg^2(X) < \infty$, then

$$Cov(g(X_{r:n}), g(X_{r+1:n})) \geq 0.$$

To prove Theorem 3.6, we set $U_{0:n} = 0$ and $U_{n+1:n} = 1$, and use the following lemma (*cf. Arnold et al., 1992, pp. 25-26*).

Lemma 3.1: For $0 < a < b < 1$, Let $U_1(a,b)$ and $U_2(a,b)$ be independent variables uniformly distributed over (a,b) and be independent of $U_1, U_2, \ldots, U_n$. Denote the order statistics of $U_1(a,b)$ and $U_2(a,b)$ as $U_{(1)}(a,b) \leq U_{(2)}(a,b)$. If $n \geq 2$ and $1 \leq r \leq n-1$, then, conditioned on $(U_{r-1:n}, U_{r+2:n})$, $(U_{r:n}, U_{r+1:n})$ has the same distribution as $(U_{(1)}(U_{r-1:n}, U_{r+2:n}), U_{(2)}(U_{r-1:n}, U_{r+2:n}))$

Proof of Theorem 3.6: For each $u, 0 < u < 1$, set

$$F^-(u) = \inf\{x : F(x) \geq u\}.$$

Note that $(F^-(U_{1:n}), F^-(U_{2:n}), \ldots, F^-(U_{n:n}))$ has the same distribution as $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$. For each u set $f(u) = g(F^-(u))$, then apply Lemma 3.1, to get

$$E\,g(X_{r:n})\,g(X_{r+1:n})$$

$$= E\,f(U_{r:n})\,f(U_{r+1:n})$$

$$= E(E(f(U_{r:n})f(U_{r+1:n})|(U_{r-1:n},U_{r+2:n})))$$

$$= E(E(f(U_{(1)}(U_{r-1:n},U_{r+2:n}))f(U_{(2)}(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n})))$$

$$= Ef(U_{(1)}(U_{r-1:n},U_{r+2:n}))f(U_{(2)}(U_{r-1:n},U_{r+2:n})) \tag{3.17}$$

and for $m = 1,2$,

$$Eg(X_{r-1+m:n}) = Ef(U_{r-1+m:n})$$

$$= E(E(f(U_{r-1+m:n})|(U_{r-1:n},U_{r+2:n})))$$

$$= E(E(f(U_{(m)}(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n})))$$

$$= Ef(U_{(m)}(U_{r-1:n},U_{r+2:n})). \tag{3.18}$$

It is obvious that

$$E(f(U_{(1)}(U_{r-1:n},U_{r+2:n})) + f(U_{(2)}(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n}))$$

$$= 2E(f(U_1(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n}))$$

and that

$$E(f(U_{(1)}(U_{r-1:n},U_{r+2:n}))f(U_{(2)}(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n}))$$

$$= (E(f(U_1(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n})))^2$$

Therefore, from (3.17) and (3.18) we get,

$$Cov(g(X_{r:n}),g(X_{r+1:n}))$$

$$= E(E(f(U_1(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n})))^2$$

$$- Ef(U_{r:n})Ef(U_{r+1:n})$$

$$= \frac{1}{4}Var(E(f(U_1(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n})))$$

$$\times \frac{1}{4}(Ef(U_{(1)}(U_{r-1:n},U_{r+2:n})) + Ef(U_{(2)}(U_{r-1:n},U_{r+2:n})))^2$$

$$- Ef(U_{r:n})Ef(U_{r+1:n})$$

$$=\frac{1}{4}Var(E(f(U_1(U_{r-1:n},U_{r+2:n}))|(U_{r-1:n},U_{r+2:n})))$$

$$+\frac{1}{4}(Ef(U_{(1)}(U_{r-1:n},U_{r+2:n}))-Ef(U_{(2)}(U_{r-1:n},U_{r+2:n})))^2$$

$$\geq 0$$

This completes the proof of Theorem 3.6.

Theorem 3.7: (*Luning Li, 1994*)

Let $X_{1:2}$ and $X_{2:2}$ denote the order statistics from a sample of size 2, and let $g(.)$ and $h(.)$ be non-decreasing functions, then

$$Cov(g(X_{1:2})h(X_{2:2}))+Cov(g(X_{2:2})h(X_{1:2}))$$

$$\leq\frac{2}{3}[Var(g(X))Var(h(X))]^{1/2}.$$

The inequality becomes equality if

$$g(x)=c_1\,h(x)=c_2\left(F(x)-\frac{1}{2}\right) \text{ for some } c_1 \text{ and } c_2.$$

Proof. By (3.1), we have

$$Cov(g(X_{1:2})-h(X_{1:2}),g(X_{2:2})-h(X_{2:2}))$$

$$=4(Cov(g(X)-h(X),F(X)))^2,$$

$$Cov(g(X_{1:2}),g(X_{2:2}))=4[Cov(g(X),F(X))]^2 \tag{3.19}$$

$$Cov(h(X_{1:2}),h(X_{2:2}))=4[Cov(h(X),F(X))]^2 . \tag{3.20}$$

Plugging (3.19) and (3.20) to the first equation, we have that

$$Cov(g(X_{1:2}),h(X_{2:2}))+Cov(g(X_{2:2}),h(X_{1:2}))$$

$$=8Cov(g(X),F(X))Cov(h(X),F(X)).$$

Using (3.3), we get

$$Cov(g(X_{1:2}),h(X_{2:2}))+Cov(g(X_{2:2}),h(X_{1:2}))$$

$$=\frac{2}{3}[Var(g(X))Var(h(X))]^{1/2}$$

which completes the proof.

3.2: Some counterexamples

The counterexamples exhibited here show that for $n \geq 3$ and each pair $1 \leq r$, $s \leq n$, if $|r-s|>1$, then one would find a function g such that $g(X_{r:n})$ and $g(X_{s:n})$ are negatively correlated.

Example 3.1: Suppose $F(x)$ is continuous in this section. Here for $r=1$ and $s=n$, where $n \geq 3$ we show that

$$Cov(g(X_{1:n}),h(X_{n:n}))<0 \qquad (3.21)$$

For, some g with $Eg^2(X)<\infty$, take $g(x)=F^2(x)-F(x)$.

Assume that $U_1,U_2,...,U_n$ are *i.i.d.* random variables uniformly distributed over (0,1) and $U_{1:n} \leq U_{2:n} \leq ... \leq U_{n:n}$ are order statistics of $U_1,U_2,...,U_n$. Since $F(X)$ is uniformly distributed over $(0,1)$, $(F(X_{1:n}),F(X_{2:n}),...,F(X_{n:n}))$ has the same distribution as $(U_{1:n} \leq U_{2:n} \leq ... \leq U_{n:n})$. Therefore to prove (3.21), it suffices to show

$$Ef(U_{1:n})f(U_{n:n})-Ef(U_{1:n})Ef(U_{n:n})<0, \qquad (3.22)$$

where $f(x)=x^2-x$. It is easily seen that $(U_{1:n},U_{n:n})$ has the joint density function, say, $p(x,y)$, where

$$p(x,y)=\begin{cases} n(n-1)(y-x)^{n-2}, & 0<x<y<1, \\ 0, & \textit{otherwise}. \end{cases}$$

Computations show that

$$Ef(U_{1:n})f(U_{n:n})-Ef(U_{1:n})Ef(U_{n:n})$$

$$=\frac{11}{3(n+1)}-\frac{3}{(n+2)}-\frac{2}{3(n+4)}-\frac{1}{(n+1)^2}-\frac{4}{(n+2)^2}$$

$$=\frac{n(2-n)}{(n+1)^2(n+2)^2(n+4)}<0.$$

The details are omitted. This proves (3.22).

Example 3.2: For $3\le n\le 10$ and $1\le r,s\le n$ with $|r-s|>1$, let

$$g_{r,s}(x)=F^2(x)-\frac{r+s}{n+1}F(x),$$

then

$$Cov(g_{r,s}(X_{r:n}),g_{r,s}(X_{s:n}))<0.$$

Example 3.3: Let $u(x)=I_{[a,b]}(x)$, $n=3$. Let X be a random variable with *df* $F(x)$. Let $F(a)=\frac{1}{4}, F(b)=\frac{3}{4}$, then

$$P(X_{1:3}\in(a,b)X_{3:3}\in(a,b))$$

$$=P(X_{1:3}\in(a,b),X_{2:3)}\in(a,b)\,,X_{3:3}\in(a,b))$$

$$=P^3(X\in(a,b))$$

$$=\frac{1}{8}$$

Similarly,

$$P(X_{1:3}\in(a,b))=\frac{26}{64}$$

and

$$P(X_{3:3}\in(a,b))=\frac{26}{64}$$

Hence

$$Cov(u(X_{1:3}),u(X_{3:3}))$$

$$=P(X_{1:3}\in(a,b),\,X_{3:3}\in(a,b))\;-P(X_{1:3}\in(a,b)\,P(X_{3:3}\in(a,b)))$$

$$=\frac{1}{8}-\left(\frac{26}{64}\right)^2=-\frac{164}{4096}<0$$

and therefore (3.1) cannot be true in general.

3.3: Upper bound for the covariance of extreme order statistics from sample of size three

Theorem 3.8: *(Papadatos, 1999)*

Let U be a uniform (0,1) random variable and suppose that $U_{1:3}, U_{3:3}$ are the minimum and maximum from a random sample of size three drawn from U. Then, for any arbitrary function $g \in L^2(0,1)$ we have

$$Cov(g(U_{1:3}), g(U_{3:3})) \le \frac{6}{a^2} Var[g(U)], \tag{3.23}$$

where $a \approx 5.96941$ is the unique positive root of the equation $\tanh(a/2) = a/6$. Equality in (3.23) holds *if* and only *if* there exists some constants B and C, such that $g(u) = C\sinh(a(u-1/2)) + B$ for almost all $u \in (0,1)$.

Corollary 3.10: Let $X_{1:3}, X_{3:3}$ be the minimum and maximum corresponding to a random sample of size three drawn from an arbitrary distribution function with mean μ and finite variance $\sigma^2 > 0$. Then,

$$Cov[X_{1:3}, X_{3:3}] \le \frac{6}{a^2}\sigma^2 \approx 0.16838\sigma^2 \tag{3.24}$$

and the equality in (3.24) characterizes the hyperbolic sine distribution with density

$$f(x) = \frac{1/a}{\sqrt{(x-\mu)^2 + \lambda^2\sigma^2}},$$

$$\mu - a\sigma\sqrt{\frac{2}{a^2-24}} < x < \mu + a\sigma\sqrt{\frac{2}{a^2-24}}$$

where $\lambda = \sqrt{2(36-a^2)/(a^2-24)} \approx 0.25089$ and a is an in Theorem 3.8.

Proof: Since $(X_{1:3}, X_{3:3}) \underline{\underline{d}} (F^{-1}(U_{1:3}), F^{-1}(U_{3:3}))$, Theorem 3.8 (with $g = F^{-1}$) shows that

$$Cov(X_{1:3}, X_{3:3}) = Cov\left\{F^{-1}(U_{1:3}), F^{-1}(U_{3:3})\right\}$$
$$\leq \frac{6}{a^2} Var\left[F^{-1}(U)\right] = \frac{6}{a^2}\sigma^2$$

which proves (3.24). Assume now that the equality holds in (3.24). From Theorem 3.8 it follows that this is equivalent to the fact that for some constants B and C.

$$F^{-1}(u) = C\sinh(a(u-1/2)) + B$$

for almost all $u \in (0,1)$. Since F is non-degenerate (because $\sigma^2 > 0$ by the assumptions), C must be non-zero. Therefore, $C > 0$ (because $F^{-1}(u)$ should be non-decreasing in u). On the other hand, since F^{-1} is always left continuous, it follows that $F^{-1}(u) = C\sinh(a(u-1/2)) + B$ for some $C > 0$ and for all $u \in (0,1)$. Observing that $F^{-1}(0_+) = B - C\sinh(a/2)$ and $F^{-1}(1_-) = B + C\sinh(a/2)$, it is concluded that F is concentrated on the finite range $|x-B| < C\sinh(a/2)$. Therefore, inverting F^{-1} we find that

$$F(x) = \frac{1}{2} + \frac{1}{a}\log\left[\frac{1}{C}\left\{ x - B + \left(C^2 + (x-B)^2\right)^{1/2}\right\}\right],$$
$$|x-B| < C\sinh(a/2).$$

and since this F is absolutely continuous, the desired result is followed by a simple calculation of the derivative of F, observing that

$$\mu = \int_0^1 F^{-1}(u)du = B$$

and

$$\sigma^2 = \int_0^1 (F^{-1}(u) - \mu)^2 du = C^2 \frac{\sinh(a) - a}{2a} = C^2 \frac{a^2 - 24}{2(36 - a^2)}.$$

4. Bounds of correlation coefficient between two order statistics

4.1: A bound for correlation between ordered sample pairs

The correlation between the ordered pairs may provide the information on the independence of the two events.

Theorem 4.1: *(Terrel, 1983)*

Let $X_{1:2}, X_{2:2}$ be an independent sample of size two from a continuous distribution $F(x)$ with finite variance. Then $X_{1:2}$ and $X_{2:2}$ are positively correlated, and $2corr(X_{1:2}, X_{2:2}) \leq 1$ with equality *if* and only *if* $F(x)$ is a rectangular distribution.

Proof: Let $F^{-1}(Y)$ be the inverse of the cumulative distribution function. It is clear from a change of variable that X having a variance is equivalent to the condition:

$$\int_0^1 (F^{-1}(Y))^2 dY$$

is finite. Thus $F^{-1}(Y)$ is a member of nondecreasing square-integrable functions on $(0,1)$. Now functions in this case may be approximated arbitrarily well by polynomials, using the L^2-metric on $(0,1)$. Let $\{L_i(Y)\}\, i = 0,1,\ldots$ be the normalized Legendre polynomials on $[0,1]$ with the properties

$$\int_0^1 L_i(Y)L_j(Y)\,dY = \delta_{ij} \qquad \text{and} \qquad L_i(1) > 0 \text{ for all } i, j.$$

Then we can expand

$$F^{-1}(Y) = \sum_{i=0}^{\infty} a_i L_i(Y)$$

where

$$\int_0^1 L_i(Y)F^{-1}(Y)\,dY = a_i$$

and thus

$$\int_0^1 F^{-1}(Y)^2\,dY = \sum_{i=0}^{\infty} a_i^2 < \infty$$

and the polynomial series converges to the L^2-metric.

Some particular properties of the Legendre's polynomials to be used to prove the given theorem are given in the following lemma,

Lemma 4.1:

(i) $$\int_{0\le X\le Y\le 1} 2L_i(X)L_i(Y)\,dX\,dY = 0, \qquad i > 0$$

(ii) $$\int_{0\le X\le Y\le 1} 2L_{i-1}(Y)L_i(X)\,dY\,dX = \frac{-1}{\sqrt{(2i-1)(2i+1)}}, \qquad i > 0$$

$$\int_{0\le X\le Y\le 1} 2L_{i-1}(X)L_i(Y)\,dX\,dY = \frac{1}{\sqrt{(2i-1)(2i+1)}}, \qquad i > 0$$

(iii) $$\int_0^1 2X\,L_i^2(X)\,dX = 1, \qquad i \ge 0$$

(iv) $$\int_0^1 2X\,L_i(X)L_{i-1}(X)\,dX = \frac{1}{\sqrt{(2i-1)(2i+1)}}, \qquad i > 0$$

Proof: (i) and (iii) follow from the alternate oddness and evenness of L_i's with respect to $\frac{1}{2}$. (ii) and (iv) follow from the observation that $\int P_i(X)L_i(X)$ (where P_i is a polynomial of degree i) depends only on the coefficient of X^i, since L_i is orthogonal to terms of lower degree.

Now let $X_{1:2}, X_{2:2}$ be a sample of size two from the distribution with quartile function F^{-1}. Then

$$corr(X_{1:2}, X_{2:2}) = \frac{Cov(X_{1:2}, X_{2:2})}{\sqrt{Var(X_{1:2})Var(X_{2:2})}}$$

$$\begin{aligned}
Cov(X_{1:2}, X_{2:2}) = & \int_{0 \le X_{1:2} \le X_{2:2} \le 1} 2F^{-1}(X_{1:2})F^{-1}(X_{2:2})dX_{1:2}\,dX_{2:2} \\
& - \int_0^1 2X_{2:2}\, F^{-1}(X_{2:2})\, dX_{2:2} \\
& \times \int_0^1 2(1 - X_{1:2})F^{-1}(X_{1:2})dX_{1:2} \\
= & \sum_{i=0}^{\infty} \int_{0 \le X_{1:2} \le X_{2:2} \le 1} 2a_i^2 L_i(X_{1:2})L_i(X_{2:2})dX_{1:2}\,dX_{2:2} \\
& + \sum_{i=0}^{\infty} \int_{0 \le X_{1:2} \le X_{2:2} \le 1} 2a_i\, a_{i+1} L_i(X_{1:2}) \\
& \times L_{i+1}(X_{2:2})dX_{1:2}\,dX_{2:2} \\
& + \sum_{i=0}^{\infty} \int_{0 \le X_{1:2} \le X_{2:2} \le 1} 2a_i\, a_{i+1} L_i(X_{2:2}) \\
& \times L_{i+1}(X_{1:2})dX_{1:2}\,dX_{2:2} \\
& - \left(a_0 + \frac{a_1}{\sqrt{3}}\right)\left(a_0 - \frac{a_1}{\sqrt{3}}\right).
\end{aligned}$$

The other cross product are zero by orthogonality of the L_i's. Now, applying the Lemma 4.1 allows us to cancel almost all terms, leaving

$$\int\limits_{0\le X_{1:2}\le X_{2:2}\le 1} 2a_1^2 L_0(X_{1:2})L_0(X_{2:2})-a_0^2+\frac{a_1^2}{3}=a_0^2-a_0^2+\frac{a_1^2}{3}=\frac{a_1^2}{3}$$

Thus, the covariance of an ordered sample of two depends only on the second (or linear) Legendre coefficient.

Now

$$Var(X_{1:2})=\int_0^1 2(1-X_{1:2})F^{-1}(X_{1:2})^2 dX_{1:2}-\left(\int_0^1 2(1-X_{1:2})F^{-1}(X_{1:2})dX_{1:2}\right)^2$$

$$=2\sum_{i=0}^{\infty}a_i^2-\sum_{i=0}^{\infty}\int_0^1 2a_i^2 X_{1:2}L_i\left(X_{1:2}\right)^2 d\,X_{1:2}$$

$$-\sum_{i=0}^{\infty}\int_0^1 4a_i\,a_{i+1}\,X_{1:2}\,L_i\left(X_{1:2}\right)L_{i+1}\left(X_{1:2}\right)dX_{1:2}$$

$$-\left(a_0-\frac{a_1}{\sqrt{3}}\right)^2$$

where once again a great many terms disappear because of the orthogonality of the L_i's to lower degree polynomials. And, once again applying the Lemma 4.1 we have

$$Var(X_{1:2})=\sum_{i=0}^{\infty}a_i^2-\sum_{i=0}^{\infty}\frac{2(i+1)a_i\,a_{i+1}}{\sqrt{(2i+1)(2i+3)}}-\left(a_0-\frac{a_1}{\sqrt{3}}\right)^2.$$

These terms can be recognized by completing squares

$$Var(X_{1:n})=\sum_{i=1}^{\infty}\left(\sqrt{\frac{i+1}{2i+1}}a_i-\sqrt{\frac{i+1}{2i+3}}a_{i+1}\right)^2.$$

Similarly

$$Var(X_{2:n}) = \sum_{i=1}^{\infty} \left(\sqrt{\frac{i+1}{2i+1}} a_i + \sqrt{\frac{i+1}{2i+3}} a_{i+1} \right)^2 .$$

Let

$$c_i = \sqrt{\frac{i+1}{2i+1}} a_i - \sqrt{\frac{i+1}{2i+3}} a_{i+1}, \qquad d_i = \sqrt{\frac{i+1}{2i+1}} a_i + \sqrt{\frac{i+1}{2i+3}} a_{i+1},$$

then

$$Var(X_{1:2})Var(X_{2:2}) = \sum_{i=1}^{\infty} c_i^2 \sum_{i=1}^{\infty} d_i^2 \geq \left(\sum_{i=1}^{\infty} c_i\, d_i \right)^2$$

by Cauchy's inequality. But

$$c_i\, d_i = \frac{i+1}{2i+1} a_i^2 - \frac{i+1}{2i+3} a_{i+1}^2$$

so

$$Var(X_{1:2})Var(X_{2:2}) \geq \left(\frac{2}{3} a_1^2 + \sum_{i=2}^{\infty} \frac{a_i^2}{2i+1} \right)^2 \geq \left(\frac{2}{3} a_1^2 \right)^2 .$$

Thus

$$corr(X_{1:2}, X_{2:2}) = \frac{Cov(X_{1:2}, X_{2:2})}{\sqrt{Var(X_{1:2})Var(X_{2:2})}} \leq \frac{a_1^2/3}{2a_1^2/3} = \frac{1}{2}.$$

Further, equality is attained precisely when $a_2^2 = a_3^2 = \cdots = a_i^2 = \cdots = 0$. It is readily seen that this is the case only for a rectangular distribution.

4.2: The best upper bound for correlation between ordered sample pairs

Theorem 4.2: *(Szekely and Mori, 1985)*

Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote their order statistics. Fix $1 \leq r < s \leq n$ and suppose that both $X_{r:n}$ and $X_{s:n}$ have finite variance. Then

$$Corr(X_{r:n}, X_{s:n}) \le \left(\frac{r(n+1-s)}{s(n+1-r)} \right)^{1/2}$$

where the equality holds if and only if the X 's are of rectangular distribution.

Remark: The best lower bound for correlations between ordered sample pairs is obviously 0. This is a simple consequence of the fact that they have increasing regression on each other. The correlation attains 0 only for degenerate distributions, but it can be arbitrarily small if the X 's take on two values, one of them with probability close to 1.

Proof: Let $F(x)$ denote the distribution function of X . Let us introduce the inverse of $F(x)$ as

$$F^{-1}(y) = \inf\{x : F(x) \ge y\}, \qquad 0 < y < 1.$$

Since $X \overset{d}{=} F^{-1}(U)$ if U is uniformly distributed on $(0,1)$. It is sufficient to deal with the maximal correlation of the ordered sample from the distribution $U\ (0,1)$. In this case the probability density function of $X_{r:n}$ and $X_{s:n}$ are

$$\varphi_r(x) = B(r, n+1-r)^{-1} x^{r-1} (1-x)^{n-r}, \qquad 0 < x < 1,$$

and

$$\varphi_s(y) = B(s, n+1-s)^{-1} y^{s-1} (1-y)^{n-s}, \qquad 0 < y < 1,$$

respectively, while their joint density function is

$$\varphi_{rs}(x,y) = B(r, s-r, n+1-s)^{-1} x^{r-1} (y-x)^{s-r-1} (1-y)^{n-s}, \qquad 0 < x < y < 1$$

where

$$B(t_1, t_2, \ldots, t_k) = \frac{\Gamma(t_1)\Gamma(t_2)\cdots\Gamma(t_k)}{\Gamma(t_1 + t_2 + \cdots + t_k)}.$$

Denote by P_k and Q_k $(k = 0,1,\ldots)$ the orthogonal polynomials on $(0,1)$ with respect to the weight function φ_r and φ_s respectively. They are simple transforms of the Jacobi polynomials $J_k^{n-r,r-1}$ and $J_k^{n-s,s-1}$, namely

$$P_k(x) = \alpha_k\, \varphi_r(x)^{-1} \frac{d^k}{dx^k}(x^k(1-x)^k \varphi_r(x))$$

$$Q_k(x) = \beta_k\, \varphi_s(x)^{-1} \frac{d^k}{dx^k}(x^k(1-x)^k \varphi_s(x))$$

where α_k and β_k are appropriate constants.

The coefficient of x^k in $P_k(x)$ is

$$p_k = \left(\frac{(2k+n)!(2k+n-1)!(r-1)!(n-r)!}{(k+n-1)!k!n!(k+r-1)!(k+n-r)!} \right)^{1/2}$$

Replacing r by s in this expression we obtain the principal coefficient of Q_k, which will be denoted by q_k.

Let f and g be square integrable functions on $(0,1)$ with respect to φ_r and φ_s respectively. Let us expand them into a series of the polynomials P_k and Q_k,

$$f = \sum_{k=0}^{\infty} a_k P_k, \qquad g = \sum_{k=0}^{\infty} b_k Q_k.$$

These series converge in the corresponding L^2-space. Clearly

$$Ef(X_{r:n}) = a_0, \qquad Eg(X_{s:n}) = b_0,$$

$$Var(f(X_{r:n})) = \sum_{k=1}^{\infty} a_k^2, \qquad Var(g(X_{s:n})) = \sum_{k=1}^{\infty} b_k^2,$$

and

$$Ef(X_{r:n})g(X_{s:n}) = \iint\limits_{0<x<y<1} f(x)g(y)\varphi_{rs}(x,y)\,dx\,dy$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_m b_m \iint_{0<x<y<1} P_k(x) Q_m(y) \varphi_{rs}(x,y)\,dx\,dy .$$

In the above double sum the terms vanish for $k \neq m$

$$\iint_{0<x<y<1} P_k(x) Q_m(y) \varphi_{rs}(x,y)\,dx\,dy$$

$$= \int_0^1 Q_m(y) \varphi_s(y) \int_0^y P_k(x) B(r,s-r)^{-1} x^{r-1} (y-x)^{s-r-1} y^{-s+1} dx\,dy$$

$$= \int_0^1 Q_m(y) \varphi_s(y) \int_0^1 P_k(ty) B(r,s-r)^{-1} t^{r-1} (1-t)^{s-r-1} dt\,dy \quad (4.1)$$

Here the inner integral is a polynomial of y of degree k, thus it is orthogonal to Q_m if $k < m$. We can proceed in a similar way for $k > m$. Hence

$$E f(X_{r:n}) g(X_{s:n}) = \sum_{k=0}^{\infty} a_k b_k c_k$$

where

$$c_k = \iint_{0<x<y<1} P_k(x) Q_k(y) \varphi_{rs}(x,y)\,dx\,dy .$$

Obviously $c_0 = 1$, thus

$$Corr(f(X_{r:n}) g(X_{s:n})) = \left(\sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} b_k^2 \right)^{-1/2} \sum_{k=1}^{\infty} a_k b_k c_k . \quad (4.2)$$

In order to calculate c_k we start from (2.1). The principal coefficient of the polynomial given by the integral is

$$p_k \frac{B(k+r,s-r)}{B(r,s-r)} = p_k \frac{(s-1)!(k+r-1)!}{(r-1)!(k+s-1)!} .$$

Hence

$$c_k = \frac{p_k (s-1)!(k+r-1)!}{q_k (r-1)!(k+s-1)!}$$

$$=\left(\frac{(s-1)!(k+n-s)!(n-r)!(k+r-1)!}{(r-1)!(k+n-r)!(n-s)!(k+s-1)!}\right)^{1/2}$$

From which

$$c_1=\left(\frac{r(n+1-s)}{s(n+1-r)}\right)^{1/2}$$

and

$$c_k=c_{k-1}\left(\frac{(k+n-s)(k+r-1)!}{(k+n-r)(k+s-1)!}\right)^{1/2}<c_{k-1}.$$

Applying the Cauchy-Schwarz inequality to (2.2) we obtain the upper bound of the Theorem. Equality holds if and only if $a_k=b_k=0$ for $(k=0,1,\ldots)$ that is if f and g are linear functions. In the case $f=g=F^{-1}$ it means that $F(x)$ is rectangular distribution.

The correlation between order statistics $X_{r:n}$ and $X_{s:n}$ may also be obtained from the results given by Khan *et al.* (2011) for generalized order statistics *(gos).*

Theorem 4.3: *(Khan et al., 2011)*

If $X(r)$ and $X(s)$ be the r^{th} and s^{th} *gos,* $1\le r<s\le n$ based on arbitrary continuous *df* $F(x)$ then

$$corr(X(r),X(s))\le\sqrt{\frac{r}{s}\frac{\gamma_s}{\gamma_r}}$$

provided $X(r)$ and $X(s)$ have finite variances and equality holds, if

$$\overline{F}(x)=(ax+b)^{1/(m+1)},\qquad \alpha<x<\beta$$

where

$$\overline{F}(x)=1-F(x).$$

This gives

$$corr(X_{r:n}, X_{s:n}) = \sqrt{\frac{r}{s}\frac{n-s+1}{n-r+1}}$$

for order statistics at $m=0$, $k=1$ and $\gamma_r = n-r+1$, if the *df* is

$$F(x) = \frac{x-\alpha}{\beta-\alpha}, \qquad \alpha < x < \beta.$$

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